Hochster’s Formula in Combinatorial Commutative Algebra

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Abstract

One of the main themes in combinatorial commutative algebra are bijections between algebraic and combinatorial objects, and how their properties coincide. This thesis studies the Stanley–Reisner correspondence, which is a bijection between squarefree monomial ideals and simplicial complexes. This relationship can be used to compute certain algebraic invariants, especially the graded Betti numbers, of squarefree monomial ideals using simplicial homology. The aim of this thesis is to present a comprehensive proof of Hochster's formula, which the most well-known closed formula for the graded Betti numbers. Hochster's formula gives the Betti numbers of a squarefree monomial ideal in terms of the dimensions of the homology groups of the corresponding simplicial complex.

Keywords Combinatorial Commutative Algebra, Simplicial Homology, Betti Numbers

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Tekniikan kandidaatintyön tiivistelmä

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Tiivistelmä

Yksi kombinatorisen kommutatiivisen algebran pääteemoista on bijektiot algebrallisten ja kombinatoristen objektien välillä, ja kuinka näiden objektien ominaisuudet vastaavat toisiaan. Tässä työssä keskitytään simpleksisten kompleksien generoimiin Stanley–Reisner ideaaleihin, jotka ovat neliövapaita monomi-ideaaleja polynomirenkaassa. Näille ideaaleille halutaan usein määrittää tiettyjä invariantteja, jotka karakterisoivat ideaalin algebrallista rakennetta. Polynomirenkaan ideaalien rakenteen tutkimiseen käytetään usein ideaalin vapaata resoluutiota, jonka perusteella voidaan määrittää ideaalin Betti-luvut. Yksi tunnetuimmista tuloksista kombinatorisessa kommutatiivisessa algebrassa on Hochsterin kaava, joka ilmaisee neliövapaiden monomi-ideaalien Betti-luvut vastaavan simpleksisen kompleksin homologiaryhmien ulottuvuuksien kautta. Työn tavoitteena on esittää kattava todistus Hochsterin kaavasta sekä tutkia eri lähestymistapoja aiheeseen.

Avainsanat Kombinatorinen kommutatiivinen algebra, Simpleksinen homologia, Betti-luvut

Contents

1 Introduction

Combinatorial commutative algebra is a relatively new and active area of research in mathematics. As its name suggests, the discipline studies the intersection of combinatorics and commutative algebra, both older and more established fields. One of the main themes in combinatorial commutative algebra are the relationships between certain combinatorial and algebraic objects, and how their properties are linked. In this thesis the bijection between simplicial complexes and squarefree monomial ideals, known as the *Stanley–Reisner correspondence*, is the link between combinatorics and commutative algebra.

One of the most studied algebraic structures in commutative algebra is the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$ over a field $\Bbbk$ in n variables, as well as modules over S, which can be thought of as generalizations of vector spaces. In order to better understand these modules, many invariants have been introduced in the literature. In this thesis the main invariant of interest are the Betti numbers of a module, which give information about the structure of the module's minimal free resolution. Despite extensive research, closed formulas for the Betti numbers of modules are rare. Hochster's formula, originally presented in [5] by Melvin Hochster, is the most well-known of these formulas, and is one of the landmark results in combinatorial commutative algebra. Hochster's formula gives the Betti numbers of a squarefree monomial ideal in terms of the dimensions of the homology groups of the corresponding simplicial complex. More explicitly, the Betti numbers of a squarefree monomial ideal I_Δ in squarefree degree $\sigma \in \mathbb{N}^n$, corresponding to the simplicial complex Δ, are given by

$$\beta_{i,\sigma}(I_\Delta) = \dim_{\Bbbk} \widetilde{H}_{i-1}(\operatorname{link}_{\Delta^\vee}(\bar{\sigma}); \Bbbk),$$

where $\widetilde{H}_{i-1}(\operatorname{link}_{\Delta^\vee}(\bar{\sigma}); \Bbbk)$ denotes the $(i-1)^{th}$ homology group of the augmented chain complex of $\operatorname{link}_{\Delta^\vee}(\bar{\sigma})$ with respect to the field $\Bbbk$.

The aim of this thesis is to present a proof of Hochster's formula while combining approaches from different sources to the theory behind the proof. Section 2 covers the basics of graded rings, modules and the polynomial ring. In Section 3 we introduce free resolutions and Betti numbers, and Section 4 covers simplicial homology and the Stanley–Reisner correspondence. Section 5 introduces tensor products of modules and the exterior algebra, and includes results from commutative algebra necessary for the proof of Hochster's formula, which is presented in Section 6.

2 Preliminaries

The basics of graded rings, modules and homomorphisms needed in this thesis are covered in this section.

Let R be a commutative ring with identity element 1. A module M over R is an abelian group together with an operation $R \times M \to M : (r, m) \mapsto r \cdot m$, for which

$$\begin{aligned} 1 \cdot m &= m \\ (r + r') \cdot m &= r \cdot m + r' \cdot m \\ r \cdot (m + m') &= r \cdot m + r \cdot m' \\ (rr') \cdot m &= r \cdot (r' \cdot m) \end{aligned}$$

for all $r, r' \in R$ and $m, m' \in M$. The operation $R \times M \to M : (r, m) \mapsto r \cdot m$ is referred to as scalar multiplication. Modules are generalizations of $\Bbbk$-vector spaces, with the ring R replacing the field $\Bbbk$. The ring R can be considered as a module over itself, and all ideals of R are submodules of R.

A subset $B \subseteq M$ is said to be basis of M if B non-empty, B generates M and B is linearly independent. A free module is a module which admits a basis. If $B \subseteq M$ is a basis of M, then M is isomorphic to the direct sum of $|B|$ copies of the ring R, i.e., $M \cong R^{|B|}$.

A commutative ring R is *graded* if it can be decomposed as a direct sum of abelian groups $R = \oplus_{i \in \mathbb{Z}} R_i$ such that $R_i R_j \subseteq R_{i+j}$. The elements of R_i are called the homogenous elements of degree i. Similarly, an R-module M is graded, if it can be decomposed as

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

into abelian groups, so that $R_i M_j \subseteq M_{i+j}$. The elements of M_i are called homogenous elements of degree i.

In this thesis the ring in question will most often be the polynomial ring in n variables over a field $\Bbbk$, which we will refer to as $S := \Bbbk[x_1, \dots, x_n]$. A monomial in S is a product $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$, with $a_i \in \mathbb{N}$ for all $i \in \{1, \dots, n\}$. The vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ is the exponent of $\mathbf{x}$. A monomial $x_1^{a_1} \dots x_n^{a_n}$ in S has total degree $a_1 + \cdots + a_n$. The *support* of a monomial $\mathbf{x}^{\mathbf{a}}$ is the set of indices of the variables that divide $\mathbf{x}^{\mathbf{a}}$

$$\operatorname{supp}(\mathbf{x}^{\mathbf{a}}) = \{i \mid x_i \text{ divides } \mathbf{x}^{\mathbf{a}}\},$$

which is equivalent to the set of indices of the variables with nonzero exponent in $\mathbf{x}^{\mathbf{a}}$.

In the case of the polynomial ring S, we consider two options for the grading. The first is the *standard grading*, where we decompose S into abelian groups of total degree i:

$$\begin{aligned} S &= \bigoplus_{i \in \mathbb{Z}} S_i \\ S_i &= (\mathbf{x}^{\mathbf{a}} \mid \deg(\mathbf{a}) = a_1 + \cdots + a_n = i\ ). \end{aligned}$$

Thus each S_i consists of homogenous polynomials of degree i, which forms a $\Bbbk$-vector space. The second option is the *multigrading*, defined as

$$S = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S_{\mathbf{a}}$$
$$S_{\mathbf{a}} = \Bbbk(\mathbf{x}^{\mathbf{a}}).$$

This is sometimes referred to as a *finer grading*, as we're considering the specific exponent vectors instead of the total degree. Here $\Bbbk(\mathbf{x}^{\mathbf{a}})$ denotes the $\Bbbk$-vector space spanned by the monomial $\mathbf{x}^{\mathbf{a}}$. Every graded R-module has a system of homogenous generators, and the degrees of the generators determine the grading of the module [7, Proposition 2.1]. We will often consider graded free modules, which are graded modules with a basis consisting of homogenous elements.

An R-module homomorphism $\varphi : M \to N$ is graded with degree i, if

$$\deg(\varphi(m)) = \deg(m) + i$$

for all homogenous $m \in M$. We are mostly interested in homomorphisms of degree 0, such that $\varphi(M_i) \subseteq N_i$, which are referred to as *graded homomorphisms*. Often our maps will be multiplication by a homogenous element of arbitrary degree, which is not in general a graded homomorphisms. In order to consider these operations as graded homomorphisms, we shift the grading in the modules we consider. An S-module M shifted by $j \in \mathbb{N}$ degrees is defined as

$$M(-j) := \bigoplus_{i\in\mathbb{N}} M(-j)_i$$

where $M(-j)_i = M_{i-j}$. Shifting a module with a finer grading is defined similarly:

$$M(-\mathbf{b}) := \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M(-\mathbf{b})_{\mathbf{a}},$$

where $M(-\mathbf{b})_{\mathbf{a}} = M_{\mathbf{a}-\mathbf{b}}$. Shifting does not change the algebraic structure of the module, but simply changes the grading.

The R-module $\mathrm{Hom}_R(M, N)$ denotes the module of homomorphisms from M to N, with the operations being pointwise addition and scalar multiplication by elements in R.

3 Resolutions and Betti numbers

The aim of this section is to introduce minimal free resolutions and Betti numbers of modules, as these are the main algebraic objects in this thesis. We also introduce monomial matrices, which are specifically aimed to describe homomorphisms when dealing with monomial ideals in the polynomial ring S.

3.1 Chain complexes

Definition 3.1. A **chain complex** $(\mathcal{C}, \varphi)$ is a sequence of R-module homomorphisms

$$\mathcal{C} : \ldots \longrightarrow M_{i+1} \xrightarrow{\varphi_{i+1}} M_i \xrightarrow{\varphi_i} M_{i-1} \longrightarrow \ldots$$

such that the image of each map is contained in the kernel of the next, so $\varphi_i \circ \varphi_{i+1} = 0$. We often refer to $(\mathcal{C}, \varphi)$ simply by $\mathcal{C}$. The homomorphims $\varphi_i : M_i \to M_{i-1}$ are referred to as *boundary maps.* A chain complex is **exact** if $\mathrm{im}(\varphi_{i+1}) = \ker(\varphi_i)$ for all homomorphisms in the sequence. The i^{th} **homology group** of $\mathcal{C}$ is defined as

$$H_i(\mathcal{C}) = \ker(d_i)/\mathrm{im}(d_{i+1}).$$

We can equivalently define a complex to be exact if $H_i(\mathcal{C}) = 0$ for all i.

Definition 3.2. A complex $(\mathbf{G}, \partial)$ is a **subcomplex** of a complex $(\mathbf{F}, \varphi)$ if $G_i \subseteq F_i$ for all i and ∂_i is the restriction of φ_i on $\mathbf{G}$.

If $(\mathbf{F}, d)$ and $(\mathbf{G}, g)$ are complexes of R-modules, then a homomorphism of complexes $\varphi : \mathbf{F} \to \mathbf{G}$ is a collection of homomorphisms $\varphi_i : F_i \to G_i$ so that $\varphi_{i-1} \circ d_i = g_i \circ \varphi_i$ for all i. Thus $\varphi : \mathbf{F} \to \mathbf{G}$ makes the following diagram commute

$$\begin{array}{cccccccc}
\mathbf{F}: & \cdots & \xrightarrow{d_{i+1}} & F_i & \xrightarrow{d_i} & F_{i-1} & \xrightarrow{d_{i-1}} & \cdots \\
 & & & \downarrow{\scriptstyle\varphi_i} & & \downarrow{\scriptstyle\varphi_{i-1}} & & \\
\mathbf{G}: & \cdots & \xrightarrow{g_{i+1}} & G_i & \xrightarrow{g_i} & G_{i-1} & \xrightarrow{g_{i-1}} & \cdots
\end{array}$$

If the complexes $\mathbf{F}$ and $\mathbf{G}$ are graded, the homomorphism of complexes $\varphi : \mathbf{F} \to \mathbf{G}$ is graded if each $\varphi_i : F_i \to G_i$ is a graded homomorphism with fixed degree. The homomorphism φ also induces a map $\varphi_i : H_i(\mathbf{F}) = \ker(d_i)/\mathrm{im}(d_{i+1}) \to \ker(g_i)/\mathrm{im}(g_{i+1}) = H_i(\mathbf{G})$ on the homology groups, as $\varphi(\ker(d_i)) \subseteq \ker(g_i)$ and $\varphi(\mathrm{im}(d_i)) \subseteq \mathrm{im}(g_i)$ [7, Lemma 3.4.]. If φ is an isomorphism, the homology groups $H_i(\mathbf{F})$ and $H_i(\mathbf{G})$ are also isomorphic [8, p. 338].

3.2 Free resolutions

Free resolutions of modules over a ring are used to characterize algebraic invariants of the module in question. Among modules over a ring, free modules are rare, as they are exactly the modules that are isomorphic to a direct sum of copies of the underlying ring. Free resolutions are a way to approximate an R-module with a sequence of free modules, which are often nice to work with in practice.

Definition 3.3. A free resolution of a finitely generated R-module M is a chain complex where all the modules in the sequence are free R-modules, such that the sequence

$$\mathbf{F}: \quad 0 \longrightarrow F_\ell \xrightarrow{d_\ell} F_{\ell-1} \xrightarrow{d_{\ell-1}} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is exact at all $i > 0$, $M \cong H_0(\mathbf{F}) = F_0/\mathrm{im}(d_1)$, and $F_i = 0$ for all $i < 0$. Sometimes for convenience, a surjective homomorphism $\epsilon : F_0 \to M$, called the *augmentation map*, is added to the resolution, such that the complex

$$\mathbf{F}: \quad 0 \longrightarrow F_\ell \xrightarrow{d_\ell} F_{\ell-1} \xrightarrow{d_{\ell-1}} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

is exact everywhere, also at F_0.

A free resolution is graded if all the boundary maps are graded (homomorphisms with degree 0), from which it follows that all modules in the sequence as well as the homology groups are graded. For a graded free resolution $\mathbf{F}$, we define $\mathbf{F}[-d]$ to be the resolution $\mathbf{F}$ shifted homologically by d degrees, so that $\mathbf{F}[-d]_i = \mathbf{F}_{i-d}$ for all $i \in \mathbb{Z}$.

In many cases it is very useful to study the components of certain degree in a graded complex. As each module of a graded complex is the direct sum of its degree components, and all boundary maps are graded, we can examine sequences of certain degree. For a graded complex $\mathbf{F}$, $(\mathbf{F})_\mathbf{a}$ denotes the degree $\mathbf{a}$ part of the complex. A graded complex is exact if and only if each of its graded components is an exact sequence of $\Bbbk$-vector spaces [7, p. 16].

Definition 3.4. A graded free resolution of an S-module is **minimal**, if $\mathrm{im}(d_i) \subseteq \mathfrak{m}F_{i-1}$ for all i, where $\mathfrak{m}F_{i-1}$ denotes the free module $(x_1, \dots, x_n)F_{i-1}$.

If we use matrices to describe our boundary maps, minimality of a resolution is equivalent to the matrices having no nonzero constants as entries. We can construct a graded free resolution of a graded finitely generated S-module M as follows. Let $m_1, \dots, m_r$ be homogenous generators of M, with degrees $a_1, \dots, a_r$ (either total degree or exponent vector). Set $F_0 = R(-a_1) \bigoplus \dots \bigoplus R(-a_r)$, with e_j being the 1-generator of $R(-a_j)$, and $\deg(e_j) = a_j$. The augmentation map surjects the generators of F_0 onto the generators of M, and is defined as

$$\begin{aligned} \epsilon : F_0 &\to M \\ e_j &\mapsto m_j, \end{aligned}$$

which is a graded homomorphism. By the first isomorphism theorem we have $F_0/\ker(\epsilon) \cong M$. If $\ker(\epsilon) = 0$, we get $F_0 \cong M$, so M is itself free. If this is not the case, next we consider $\ker(\epsilon)$ and its homogenous generators $r_1, \dots, r_k$, with degrees $h_1, \dots, h_k$. Set $F_1 = S(-h_1) \oplus \dots \oplus S(-h_k)$, with g_j being the 1-generator of $S(-h_j)$, and $\deg(g_j) = h_j$. The map d_1 is defined as

$$\begin{aligned} d_1 : F_1 &\to \ker(\epsilon) \subset F_0 \\ g_j &\mapsto r_j, \end{aligned}$$

which is a surjective graded homomorphism. We continue this process, until we get an injective map, which happens when $\ker(d_i) = 0$. The resulting free resolution of M depends on the choice of generators at each point. If we choose a minimal set of homogenous generators (with respect to cardinality) for the kernel at each step of the construction, we are guaranteed to get a minimal free resolution [7, Theorem 7.3.]. Hilbert's Syzygy Theorem states that the minimal graded free resolution of a graded finitely generated S-module has at most length n, where n is the number of variables in the polynomial ring S [7, Theorem 15.2.], and thus the minimal free resolution of a module M is finite. In addition, minimal free resolutions of an S-module are unique up to isomorphism [1, Theorem 20.2].

Example 3.5. Consider the ideal $I = (xy^2, y^3z^2)$ of the ring $S = k[x, y, z]$. Computation with Macaulay2 [3] shows that the minimal free resolution of S/I is given by

$$0 \longrightarrow S(xy^3z^2) \xrightarrow{\begin{bmatrix} -yz^2 \\ x \end{bmatrix}} S(xy^2) \oplus S(y^3z^2) \xrightarrow{\begin{bmatrix} xy^2 & y^3z^2 \end{bmatrix}} S \longrightarrow S/I \longrightarrow 0,$$

where $S(\mathbf{x}^{\mathbf{a}})$ denotes the polynomial ring shifted by $-\mathbf{a}$.

3.3 Betti numbers

For a minimal free resolution of a graded module M

$$\mathbf{F}: \quad 0 \longrightarrow F_\ell \xrightarrow{d_\ell} F_{\ell-1} \xrightarrow{d_{\ell-1}} \ldots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \longrightarrow 0,$$

the decompositions of the free modules F_i are written as

$$F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \qquad \text{or} \qquad F_i = \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{i,j}}$$

in the $\mathbb{N}^n$-graded case and in the $\mathbb{N}$-graded case, respectively. The numbers $\beta_{i,j}$ are the $\mathbb{N}$-graded Betti numbers, and $\beta_{i,\mathbf{a}}$ the $\mathbb{N}^n$-graded Betti numbers. The $\mathbb{N}^n$ and $\mathbb{N}$ graded Betti numbers are related by

$$\beta_{i,j} = \sum_{\deg(\mathbf{a})=j} \beta_{i,\mathbf{a}}.$$

Sometimes we consider the i^{th} *total Betti number* β_i, where we disregard the specific shifts of S in the decomposition of F_i:

$$\beta_i = \sum_{j \in \mathbb{Z}} \beta_{i,j}.$$

In the case of ideals in a polynomial ring, we can interchangeably talk about minimal free resolutions and Betti numbers of I and S/I due to the following lemma.

Lemma 3.6. For an ideal $I \subseteq S$ and $\mathbf{a} \in \mathbb{N}^n$,

$$\beta_{i,\mathbf{a}}(I) = \beta_{i+1,\mathbf{a}}(S/I)$$

Proof. From a minimal free resolution $\mathbf{F}$ of I,

$$\mathbf{F}: \quad 0 \longrightarrow F_\ell \longrightarrow \dots \longrightarrow F_0 \longrightarrow I \longrightarrow 0,$$

we can construct a minimal free resolution $\mathbf{F}'$ of S/I, given by

$$\mathbf{F}': \quad 0 \longrightarrow F_\ell \longrightarrow \dots \longrightarrow F_0 \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

The map $S \to S/I$ is the projection map, and otherwise the boundary maps of $\mathbf{F}'$ are equal to those of $\mathbf{F}$. $\square$

Example 3.7. Consider $I = (xy^2, y^3z^2) \subset S = k[x, y, z]$, the ideal from Example 3.5. From the minimal free resolution of S/I, we see that the multigraded Betti numbers of S/I are

$$\begin{aligned}\beta_{1,xy^2} &= 1\\ \beta_{1,y^3z^2} &= 1\\ \beta_{2,xy^3z} &= 1,\end{aligned}$$

and 0 for all other $\beta_{i,\mathbf{a}}$.

3.4 Monomial matrices

As we are focusing on squarefree monomial ideals in the multigraded case, we can use *monomial matrices*, introduced in [6], to represent the boundary maps efficiently for our purposes. One of the main advantages of monomial matrices is that they give an explicit relationship between the minimal free resolutions of monomial ideals and reduced chain complexes of certain simplicial complexes, which we will make use of in later sections. In the following we denote $\mathbf{a} \succeq \mathbf{b}$ for exponents $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ if $a_i \geq b_i$ for all $i \in [n]$. Thus $\mathbf{x}^\mathbf{b}$ divides $\mathbf{x}^\mathbf{a}$ if and only if $\mathbf{a} \succeq \mathbf{b}$.

Definition 3.8. A **monomial matrix** is an matrix of scalar entries λ_{qp} whose columns are labeled by source degree $\mathbf{a}_p$ and rows by target degree $\mathbf{a}_q$, where the entry $\lambda_{qp} \in \Bbbk$ is zero unless $\mathbf{a}_p \succeq \mathbf{a}_q$.

In general, boundary maps described by monomial matrices look like

$$\bigoplus_p S(-\mathbf{a}_p) \xrightarrow{\begin{array}{c|c} & \cdots\ \mathbf{a}_p\ \cdots \\ \hline \begin{matrix}\vdots\\ \mathbf{a}_q\\ \vdots\end{matrix} & \begin{bmatrix} & & \\ & \lambda_{qp} & \\ & & \end{bmatrix}\end{array}} \bigoplus_q S(-\mathbf{a}_q).$$

The rows and columns are labeled with monomials $\mathbf{x}^{\mathbf{a}}$ or exponent vectors $\mathbf{a}$. The entry λ_{qp} indicates that the basis of $S(-\mathbf{a}_p)$ maps to the monomial that is $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}$ times the basis of $S(-\mathbf{a}_q)$, and has coefficient λ_{qp}. The usual notation of boundary maps can be recovered by replacing the entry λ_{qp} by $\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q}$ in the matrix. A monomial matrix is minimal if $\lambda_{qp} = 0$ when $\mathbf{a}_p = \mathbf{a}_q$. A free resolution is minimal if all boundary maps are minimal monomial matrices.

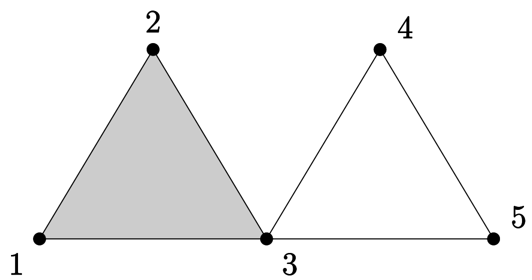
4 Simplicial complexes and homology

This section introduces simplicial complexes, their reduced chain complexes and homological properties, and the relation between squarefree monomial ideals and simplicial complexes.

4.1 Basic definitions

Definition 4.1. Let $[n]$ denote the set $\{1, \dots, n\}$. A **simplicial complex** Δ, defined on the vertex set $[n]$, is a collection of subsets of $[n]$, called faces, such that for every $X \in \Delta$, $Y \subseteq X$ implies that $Y \in \Delta$. A face with cardinality $|X| = i+1$ has dimension i, and is called an i-face of Δ. The dimension of Δ is the maximum of the dimension of the faces of Δ. The complement of a face σ is denoted by $\bar{\sigma} = \{1, \dots, n\} \setminus \sigma$.

Example 4.2. A simplicial complex Δ on the vertex set [5], consisting of the faces $\{1,2,3\}, \{2,4\}, \{4,5\}, \{2,5\}$ and their subsets, pictured below:



Definition 4.3. The **restriction** of Δ to $V \subseteq [n]$ is the complex

$$\Delta|_V = \{F \in \Delta \mid F \subset V\}.$$

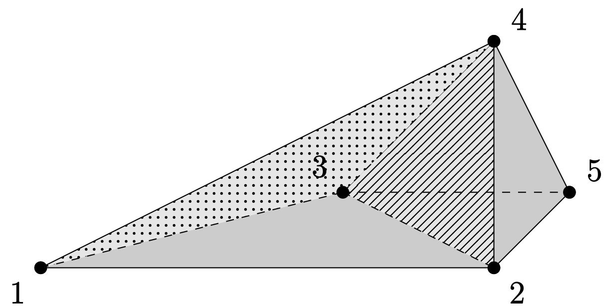
We will at times consider the *Alexander dual* of a simplicial complex, which consists of the complements of the non-faces of a simplicial complex:

Definition 4.4. The **Alexander dual** of a simplicial complex Δ on $[n]$ is

$$\Delta^\vee = \{[n] \setminus \sigma \mid \sigma \notin \Delta\}.$$

This is a duality in the sense that $(\Delta^\vee)^\vee = \Delta$, as complements of the non-faces of $\Delta^\vee$ are the faces of Δ.

Example 4.5. The Alexander dual of Δ in 4.2, with the non-faces $\{1,3,4\}$ and $\{2,3,4\}$ patterned, is given by



Definition 4.6. The **link** of a face σ of Δ is the set of faces of Δ whose union with σ is a face of Δ and that are disjoint from σ:

$$\mathrm{link}_\Delta \sigma = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta,\ \sigma \cap \tau = \emptyset\}.$$

4.2 Stanley–Reisner correspondence

There is a one-to-one correspondence between simplicial complexes and ideals generated by squarefree monomials. This relation is know as the *Stanley–Reisner correspondence*, and it allows us to compute algebraic properties of squarefree monomial ideals using the corresponding simplicial complexes and their homological properties.

For a set $\tau \subseteq [n]$, the corresponding squarefree monomial has exponent vector with entry 1 in the i^{th} position if $i \in \tau$, and 0 otherwise. The monomial corresponding to $\tau \subseteq [n]$ is thus $\mathbf{x}^\tau = \prod_{i \in \tau} x_i$.

Definition 4.7. The **Stanley–Reisner ideal** of a simplicial complex Δ on $[n]$ is the squarefree monomial ideal generated by the non-faces of Δ

$$I_\Delta = (x^\tau | \tau \notin \Delta)$$

in the polynomial ring S in n variables. This ideal may also be defined as the intersection of monomial prime ideals. We denote $\mathfrak{m}^\tau = (x_i \mid i \in \tau)$ for the monomial prime ideal corresponding to τ. The Stanley–Reisner ideal of Δ is can equivalently be defined as

$$I_\Delta = \cap_{\tau \in \Delta} \mathfrak{m}^{\bar{\tau}}.$$

The **Stanley–Reisner ring** of Δ is the quotient S/I_Δ.

Definition 4.8. For a squarefree monomial ideal $I = (x^{\sigma_1}, \ldots, x^{\sigma_r})$, the **squarefree Alexander dual** of I is

$$I^\vee = \mathfrak{m}^{\sigma_1} \cap \ldots \cap \mathfrak{m}^{\sigma_r}.$$

This notion of Alexander duality for ideals corresponds with simplicial Alexander duality by $I_{\Delta^\vee} = I_\Delta^\vee$ [6, p. 16].

Example 4.9. The Stanley–Reisner ideal of Δ from 4.2 is

$$I_\Delta = (x_1x_4, x_2x_4, x_1x_5, x_2x_5, x_3x_4x_5).$$

4.3 Simplicial homology

In order to apply algebraic methods on simplicial complexes, we construct chain complexes from the underlying simplicial complex. Let Δ be a simplicial complex on $[n]$, and $F_i(\Delta)$ be the set of i-faces of Δ. Let $\Bbbk^{F_i(\Delta)}$ be a vector space over $\Bbbk$ with basis elements e_σ corresponding to the i-faces of Δ. Then, the reduced chain complex $\mathcal{C}_\circ(\Delta, \Bbbk)$ of Δ over $\Bbbk$ is

$$\mathcal{C}_\circ(\Delta, \Bbbk) : 0 \longrightarrow \Bbbk^{F_{n-1}(\Delta)} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_{i+1}} \Bbbk^{F_{i+1}(\Delta)} \xrightarrow{\partial_{i+1}} \ldots \xrightarrow{\partial_0} \Bbbk^{F_{-1}(\Delta)} \longrightarrow 0,$$

with the boundary maps defined as

$$\partial_i(e_\sigma) = \sum_{j\in\sigma} \operatorname{sign}(j,\sigma)e_{\sigma\setminus j},$$

where $\operatorname{sign}(j,\sigma) = (-1)^{r-1}$ when j is the rth element of the set $\sigma \subseteq [n]$. For example, if Δ is a simplicial complex on [5] with 2-face $\{1,2,3\}$, then $\partial_2(e_{\{1,2,3\}}) = e_{\{2,3\}} - e_{\{1,3\}} + e_{\{1,2\}}$.

When vectors in $\Bbbk^{F_i(\Delta)}$ are viewed as columns of length $|F_i(\Delta)|$, the boundary map ∂_i can be represented as a matrix of dimension $F_i(\Delta) \times F_{i-1}(\Delta)$. The $\Bbbk$-vector space

$$\widetilde{H}_i = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$$

is the i^{th} homology group of the reduced chain complex.

Elements of $\ker(\partial_i)$ are called i-cycles and elements of $\operatorname{im}(\partial_{i+1})$ are called i-boundaries. Loosely speaking, the i^{th} homology group of a simplicial complex Δ can be viewed as an algebraic measure of the number of i-dimensional holes of Δ.

Example 4.10. The reduced chain complex of Δ from 4.2 is given by

$$0 \longrightarrow \Bbbk \xrightarrow{\begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \Bbbk^6 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}} \Bbbk^5 \xrightarrow{\begin{bmatrix} -1 & -1 & -1 & -1 & -1 \end{bmatrix}} \Bbbk \longrightarrow 0.$$

We will also present Hochster's original result through cohomology. For a $\Bbbk$-vector space V, we denote V^* for the vector space dual $\operatorname{Hom}_\Bbbk(V,\Bbbk)$, which is a vector space consisting of all linear maps from V to $\Bbbk$. Given a basis $\{e_1,\dots,e_k\}$ of V, there is a corresponding dual basis $\{e_1^*,\dots,e_k^*\}$ of V^*, which is defined as $e_i^*(c_1e_1+\cdots+c_ke_k) = c_i$, with $c_j \in \Bbbk$ for $j = 1,\dots,k$.

Definition 4.11. The **reduced cochain complex** of Δ over $\Bbbk$ is the vector space dual $\mathcal{C}^\circ(\Delta;\Bbbk) = (\mathcal{C}_\circ(\Delta;\Bbbk))^*$ of the reduced chain complex of Δ. Each vector space $\Bbbk^{F_i^*(\Delta)} = (\Bbbk^{F_i(\Delta)})^*$ has basis $F_i^*(\Delta) = \{e_\sigma^* \mid \sigma \in F_i(\Delta)\}$, which is dual to the basis $\{e_\sigma \mid \sigma \in F_i(\Delta)\}$ of $\Bbbk^{F_i(\Delta)}$. The **coboundary map** ∂^i for an $(i-1)$-face σ is

$$\partial^i(e_\sigma^*) = \sum_{\substack{j\notin\sigma \\ j\cup\sigma\in\Delta}} \operatorname{sign}(j,\sigma\cup j)e^*_{\sigma\cup j}.$$

Using matrices to describe the coboundary maps, each ∂^i is the transpose of the boundary map ∂_i [6, p. 10]. The $\Bbbk$-vector space

$$\widetilde{H}^i = \ker(\partial^i)/\operatorname{im}(\partial^{i+1})$$

is the i^{th} cohomology group of the cochain complex.

Theorem 4.12 (**Alexander duality**)**.** Let Δ be a simplicial complex on $[n]$. Then the homology and cohomology groups are related by

$$\widetilde{H}_{i-1}(\Delta^{\vee}; \Bbbk) \cong \widetilde{H}^{n-2-i}(\Delta; \Bbbk)$$

See [6, Theorem 5.6] for a proof of this theorem.

When computing the homology groups of a simplicial complex, we can sometimes use general properties of simplicial complexes to determine the homology of the chain complex. For our purposes, properties of cones are very useful, as the homology groups will all be 0. Cones are defined as follows.

Definition 4.13. Let Δ be a simplicial complex, and k a vertex not in Δ. The **cone** of Δ with **apex** k is the simplicial complex obtained by adding $\tau \cup \{k\}$ to Δ for every $\tau \in \Delta$. Equivalently, the complex Δ is a cone with apex k if $\sigma \cup \{k\}$ is a face of Δ whenever σ is a face of Δ.

Proposition 4.14. If Δ is a cone with apex k, then

$$\widetilde{H}_i(\Delta, \Bbbk) = 0$$

for all i. We refer to [2, p. 1853] for details.

5 Algebraic machinery

The aim of this section is to cover the necessary algebraic techniques to present the proof of Hochster's formula. Tensor products and the exterior algebra of modules are introduced, and then used to construct the minimal free resolution of the residue field $\Bbbk = S/\mathfrak{m}$, which is the Koszul complex of the indeterminates of S. Some important results in this section, such as Lemma 5.3 and Theorem 5.4, will be used directly in the proof of Hochster's formula.

5.1 Tensor products and Tor

Let M, N and P be R-modules. A map $\varphi : M \times N \to P$ is said to be R-bilinear, if for each $m \in M$ and $n \in N$ the maps $n \mapsto \varphi(m, n)$ and $m \mapsto \varphi(m, n)$ are R-linear, and $\varphi(m, rn) = \varphi(rm, n)$ for all $r \in R$. Let F be the free R-module defined as $F = \bigoplus_{(m,n)\in M\times N} R$. The elements of F are linear combinations of elements in $M \times N$ with coefficients in R. Let D be the submodule generated by the elements

$$\begin{aligned}
&(m + m', n) - (m, n) - (m', n),\\
&(m, n + n') - (m, n) - (m, n'),\\
&(rm, n) - r(m, n),\\
&(m, rn) - r(m, n),
\end{aligned}$$

where $m, m' \in M, n, n' \in N, r \in R$. The tensor product of M and N over R is an R-module, denoted by $M \bigotimes_R N$, and is defined as

$$M \bigotimes\nolimits_R N := F/D.$$

For each element $(m, n) \in M \times N$, $m \otimes n$ denotes its image in $M \bigotimes_R N$. We leave the subscript in $\bigotimes_R$ out when the ring is clear from context. Elements of the form $m \otimes n$ are said to be *elementary tensors*. If $\{m_i\}_{i\in I}$ and $\{n_j\}_{j\in J}$ are spanning sets of M and N, the tensor product $M \bigotimes N$ is linearly spanned by elementary tensors $m_i \otimes n_j$.

This construction satisfies the following universal property: the bilinear map $\bigotimes : M \times N \to M \bigotimes N$ is constructed in such a way that for each bilinear map $B : M \times N \to P$ there exists a unique linear map $T : M \bigotimes N \to P$ making the following diagram commute.

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"] \arrow[dr, "B"'] & M \bigotimes N \arrow[d, dashed, "\exists!\, T"] \\
& P
\end{tikzcd}$$

The tensor product of graded modules M, N is also graded. The degree $\mathbf{c}$ component of $M \bigotimes N$ is generated by elementary tensors of the form $m_{\mathbf{a}} \otimes n_{\mathbf{b}}$, such that $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

The tensor product of multiple R-modules $M_1, \ldots, M_k$ is constructed by taking tensor products of two modules at a time. The k^{th} *tensor power* of M is defined as

$$M^{\otimes k} = M \bigotimes \ldots \bigotimes M,$$

with k copies of M on the right-hand side. This module is linearly spanned by the elementary tensors $m_1 \otimes \cdots \otimes m_k$.

Lemma 5.1. Let M be an R-module and $\mathfrak{a}$ an ideal of R. Then

$$R/\mathfrak{a} \bigotimes M \cong M/\mathfrak{a}M \qquad \text{and} \qquad M \bigotimes R \cong M.$$

Proof. By the universal property, the bilinear map from $R/\mathfrak{a} \times M$ to $M/\mathfrak{a}M$ defined as

$$\begin{aligned} f : R/\mathfrak{a} \times M &\to M/\mathfrak{a}M \\ f(r, m) &\mapsto \overline{rm} \end{aligned}$$

induces a linear map from $R/\mathfrak{a} \bigotimes M$ to $M/\mathfrak{a}M$, and this map can be verified to be bijective. To construct an inverse, we have a well-defined linear map from M to $R/\mathfrak{a} \bigotimes M$ given by

$$m \mapsto \overline{1} \otimes m,$$

such that $\mathfrak{a}M$ is in the kernel of this map. Thus we have a homomorphism from $M/\mathfrak{a}M$ to $R/\mathfrak{a} \bigotimes M$, which is the inverse of our earlier map. For the ideal $\mathfrak{a} = 0$ we have $M \bigotimes R \cong M$. □

For R-module homomorphisms $f : M \to M'$ and $g : N \to N'$, the tensor product $f \otimes g$ is defined as

$$\begin{aligned} f \otimes g : M \bigotimes N &\to M' \bigotimes N' \\ m \otimes n &\mapsto f(m) \otimes g(n). \end{aligned}$$

Definition 5.2. Let M, N be R-modules, and $\mathbf{F}$ a resolution of M. The complex $\mathbf{F} \bigotimes_R N$ is given by

$$\mathbf{F} \bigotimes_R N : \ldots \longrightarrow F_i \bigotimes N \xrightarrow{d_i \otimes 1} F_{i-1} \bigotimes N \xrightarrow{d_{i-1} \otimes 1} \ldots \xrightarrow{\epsilon \otimes 1} M \bigotimes N \longrightarrow 0,$$

where 1 denotes the identity map of N. This complex is not in general exact. If tensoring a resolution with N preserves exactness, N is said to be *flat*. The homology groups of the above complex are defined as

$$\mathrm{Tor}_i^R(M, N) := H_i(\mathbf{F} \bigotimes_R N).$$

Lemma 5.3. Let M be a $\mathbb{N}^n$ graded S-module and $\Bbbk = S/\mathfrak{m}$ the residue field. Then

$$\beta_{i,\mathbf{a}} = \dim_k \mathrm{Tor}_i^S(M, \Bbbk)_{\mathbf{a}}.$$

Proof. Let $\mathbf{F}$ be a minimal free resolution of M. Tensoring the minimal free resolution of M with $\Bbbk$ yields the resolution

$$\mathbf{F} \bigotimes_S \Bbbk : \ F_\ell \bigotimes \Bbbk \xrightarrow{d_\ell \otimes 1} F_{\ell-1} \bigotimes \Bbbk \xrightarrow{d_{\ell-1} \otimes 1} \cdots \xrightarrow{d_1 \otimes 1} F_0 \bigotimes \Bbbk \xrightarrow{\epsilon \otimes 1} M \bigotimes \Bbbk \longrightarrow 0.$$

As $\mathbf{F}$ is a minimal resolution, $\operatorname{im}(d_i) \subseteq \mathfrak{m}F_{i-1}$. The boundary maps will all be the zero map, as

$$\begin{aligned}(d_i \otimes 1)(f \otimes \overline{a}) &= d_i(f) \otimes \overline{a} = \mathbf{x}^{\mathbf{a}} f' \otimes \overline{a} \\ &= f' \otimes \mathbf{x}^{\mathbf{a}} \overline{a} = f' \otimes \overline{\mathbf{x}^{\mathbf{a}} a} = f' \otimes 0 = 0\end{aligned}$$

for some $\mathbf{a} \in \mathbb{N}^n, \mathbf{a} \neq (0, \ldots, 0)$. By Lemma 5.1, $F_i \bigotimes (S/\mathfrak{m}) \cong F_i/\mathfrak{m}F_i$, and thus the homology groups are

$$H_i(\mathbf{F} \bigotimes_S \Bbbk) = \ker(d_i \otimes 1)/\operatorname{im}(d_{i-1} \otimes 1) = F_i \bigotimes_S \Bbbk/0 \cong F_i/\mathfrak{m}F_i \cong \bigoplus_{\mathbf{a} \in \mathbb{N}^n} \Bbbk(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$$

from which it follows that $\beta_{i,\mathbf{a}} = \dim_\Bbbk \operatorname{Tor}_i^S(M, \Bbbk)_{\mathbf{a}}$. □

Theorem 5.4. Let M and N be R-modules. Then

$$\operatorname{Tor}_i^R(N, M) \cong \operatorname{Tor}_i^R(M, N).$$

We present a sketch of the proof based on [6, Exercise 1.12]. See [8, Theorem 7.1] for a more detailed proof.

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be free resolutions of M and N, respectively, with boundary maps ϕ and φ. Let $\mathcal{F} \otimes \mathcal{G}$ denote the free module $\oplus_{i,j} F_i \otimes G_i$. Arrange the summands in a rectangular array, with $F_i \oplus G_j$ at row i and column j:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & F_{i+1} \otimes G_{j+1} \arrow[r] \arrow[d] & F_{i+1} \otimes G_j \arrow[r] \arrow[d] & F_{i+1} \otimes G_{j-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F_i \otimes G_{j+1} \arrow[r] \arrow[d] & F_i \otimes G_j \arrow[r] \arrow[d] & F_i \otimes G_{j-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F_{i-1} \otimes G_{j+1} \arrow[r] \arrow[d] & F_{i-1} \otimes G_j \arrow[r] \arrow[d] & F_{i-1} \otimes G_{j-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Equip each row of this diagram with a boundary map $(-1)^i \otimes \varphi_j$. As $(F_i \bigotimes G_0)/(F_i \bigotimes \operatorname{im}(\varphi_1)) \cong F_i \bigotimes (G_0/\operatorname{im}(\varphi_1)) \cong F_i \bigotimes N$, the first homology group of this row is isomorphic to $F_i \bigotimes N$. Furthermore, the modules are given by $F_i \bigotimes G_j \cong S^r \bigotimes G_j \cong (S \bigotimes G_j)^r \cong G_j^r$, with the maps simplifying to $\varphi_j' : G_j^r \to G_{j-1}^r$. As $\ker(\varphi_{j-1}) = \operatorname{im}(\varphi_j)$, the rows are also exact, so the i^{th} row is a free resolutions of $F_i \bigotimes N$.

Next, we define a *total differential* ∂ on $\mathcal{F} \otimes \mathcal{G}$ by $\partial(f \otimes g) = \phi_i(f) \otimes g + (-1)^i f \otimes \varphi_j(g)$. This satisfies $\partial \circ \partial = 0$, as

$$\begin{aligned}
\partial(\phi_i(f) \otimes g + (-1)^i f \otimes \varphi_j(g)) &= (\phi_{i-1} \circ \phi_i)(f) \otimes g + (-1)^{i-1}\phi_i(f) \otimes \varphi_j(g) \\
&\quad + (-1)^i \phi_i(f) \otimes \varphi_j(g) - f \otimes (\varphi_{j-1} \circ \varphi_j)(g) \\
&= (-1)^{i-1}\phi_i(f) \otimes \varphi_j(g) + (-1)^i \phi_i(f) \otimes \varphi_j(g) \\
&= 0.
\end{aligned}$$

From this we get a *total complex* $\operatorname{tot}(\mathcal{F} \otimes \mathcal{G})$ by setting $\operatorname{tot}(\mathcal{F} \otimes \mathcal{G})_k = \oplus_{i+j=k} F_i \otimes G_j$ in homological degree k. In the above diagram, the summands for each k lie on the upward diagonals. Next, we define the map $\mathcal{F} \otimes \mathcal{G} \to \mathcal{F} \otimes N$ which kills $F_i \otimes G_j$ for $j > 0$ and maps $F_i \otimes G_0 \twoheadrightarrow F_i \otimes N$. This induces a morphism of complexes $\operatorname{tot}(\mathcal{F} \otimes \mathcal{G}) \to \mathcal{F} \otimes N$, with the i^{th} differential of $\mathcal{F} \otimes N$ being the map $\phi_i \otimes 1$. It can be verified that the morphism $\operatorname{tot}(\mathcal{F} \otimes \mathcal{G}) \to \mathcal{F} \otimes N$ induces an isomorphism of homology, so $H_i(\operatorname{tot}(\mathcal{F} \otimes \mathcal{G})) \cong H_i(\mathcal{F} \otimes N) = \operatorname{Tor}_i^S(M, N)$.

The above argument can be transposed by instead considering the vertical differential $\phi \otimes 1$, which makes each column into a free resolution of $M \otimes \mathcal{G}$. The map $\mathcal{F} \otimes \mathcal{G} \to M \otimes \mathcal{G}$ that kills $F_i \otimes G_j$ for $i > 0$ and maps $F_0 \otimes G_j \twoheadrightarrow M \otimes G_j$ induces a morphism of complexes $\operatorname{tot}(\mathcal{F} \otimes \mathcal{G}) \to M \otimes \mathcal{G}$, and as above, an isomorphism of homology $H_i(\operatorname{tot}(\mathcal{F} \otimes \mathcal{G})) \cong H_i(M \otimes \mathcal{G}) = \operatorname{Tor}_i^S(N, M)$. It follows that $\operatorname{Tor}_i^S(M, N) \cong H_i(\operatorname{tot}(\mathcal{F} \otimes \mathcal{G})) \cong \operatorname{Tor}_i^S(N, M)$. $\square$

5.2 The exterior algebra and Koszul complex

Next, we will introduce the exterior algebra of a module, and define the Koszul complex of a sequence of elements of the underlying ring. The aim of this section is to describe the minimal free resolution of $\mathbb{k} = S/\mathfrak{m}$, which we will use to relate certain algebraic properties of minimal free resolutions of squarefree monomial ideals to simplicial homology.

Definition 5.5. Let M be an R-module and T_k the submodule of $M^{\otimes k}$ spanned by elementary tensors of the form $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some $i \neq j$. The k^{th} **exterior power** $\Lambda^k(M)$ is the quotient

$$\Lambda^k(M) = M^{\otimes k}/T_k,$$

with $\Lambda^0(M) = R$ and $\Lambda^1(M) = M$.

The coset of $m_1 \otimes \cdots \otimes m_k$ in $\Lambda^k(M)$ is denoted by the wedge product $m_1 \wedge \cdots \wedge m_k$. If M is free and finitely generated by $m_1, \ldots, m_d$, $\Lambda^k(M)$ is spanned as an R-module by the $\binom{d}{k}$ elementary wedge products $m_{i_1} \wedge \cdots \wedge m_{i_k}$, and thus has rank $\binom{d}{k}$, as the elemetary wedge product $m_1 \wedge \cdots \wedge m_k$ will have $m_i = m_j$ for some $i \neq j$ when $k > d$.

Definition 5.6. The **Exterior algebra** of M is the direct sum

$$\Lambda(M) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Lambda^k(M).$$

The exterior algebra of M is a graded algebra, with $\Lambda^i(M)$ being the degree i component. If the free module M has rank d, so $M \cong S^d$, then the exterior algebra has rank 2^d.

Definition 5.7. Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements of R, and F the free module with basis $e_1, \ldots, e_m$. The **Koszul complex** $\mathbf{K}(\mathbf{f})$ of the sequence $\mathbf{f}$ is a chain complex consisting of free R-modules $K_j = \Lambda^j F$ with K_j having basis $\{e_{i_1} \wedge \ldots \wedge e_{i_j} \mid 1 \leq i_1 < \ldots < i_j \leq m\}$, and boundary maps defined as

$$\partial_j : K_j = \Lambda^j F \to \Lambda^{j-1} F = K_{j-1}$$
$$\partial_j(e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^{k+1} f_{i_k} e_{i_1} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_j},$$

which satisfies $\partial_i \circ \partial_{i+1} = 0$. The Koszul complex of $f_1, \ldots, f_m$ is of the form

$$\mathbf{K}(\mathbf{f}) : 0 \longrightarrow K_m \xrightarrow{\partial_m} K_{m-1} \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \longrightarrow 0.$$

From the basis sets of the free modules K_j we see that the rank of K_j is $\binom{m}{j}$. Each free module K_j is graded by $\deg(e_{i_1} \wedge \ldots \wedge e_{i_j}) = j$.

Definition 5.8. Let M be a finitely generated R-module. The **Koszul homology** of a sequence $\mathbf{f}$ with respect to M is the homology of the complex

$$\mathbf{K}(\mathbf{f}; M) := \mathbf{K}(\mathbf{f}) \bigotimes_R M.$$

The chain complex $\mathbf{K}(\mathbf{f}; M)$ is not necessarily exact. Exactness occures when $\mathbf{f}$ is a *regular* sequence, which is defined as follows.

Definition 5.9. Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements of R and M an R-module. The sequence $\mathbf{f}$ is M**-regular** if $(f_1, \ldots, f_m)M \neq M$ and f_k is a nonzero divisor in the quotient $M/(f_1, \ldots, f_{k-1})$ for all $k \in \{1, \ldots, m\}$.

Lemma 5.10. *Let $\mathbf{f} = f_1, \ldots, f_r$ be a sequence of elements in R, and $\mathbf{f}' = f_1, \ldots, f_{r-1}$. Then there is an exact sequence of complexes*

$$0 \longrightarrow \mathbf{K}(\mathbf{f}') \longrightarrow \mathbf{K}(\mathbf{f}) \longrightarrow \mathbf{K}(\mathbf{f}')[-1] \longrightarrow 0,$$

where $\mathbf{K}(\mathbf{f}')[-1]$ denotes the complex $\mathbf{K}(\mathbf{f}')$ shifted by -1 in homological degree. In addition, the sequence

$$\ldots \longrightarrow H_i(\mathbf{K}(\mathbf{f}')) \longrightarrow H_i(\mathbf{K}(\mathbf{f})) \longrightarrow H_{i-1}(\mathbf{K}(\mathbf{f}')) \xrightarrow{(-1)^{i+1} f_r} H_{i-1}(\mathbf{K}(\mathbf{f}')) \longrightarrow \ldots$$

of homology groups, where the map $(-1)^{i+1} f_r$ denotes multiplication by $(-1)^{i+1} f_r$, is exact.

See [7, p. 50-51] for a proof of this lemma.

Theorem 5.11. Let M be a finitely generated graded R-module and $\mathbf{f} = f_1, \ldots, f_r$ a sequence of homogenous elements of R with $\deg(f_i) > 0$ for all $i \in \{1, \ldots, r\}$. If $\mathbf{f}$ is an M-regular sequence, then

$$H_i(\mathbf{K}(\mathbf{f}; M)) = 0 \; \forall i > 0 \quad \text{and} \quad H_0(\mathbf{K}(\mathbf{f}; W)) = M/(\mathbf{f})M.$$

Proof. From the construction of Koszul homology, $H_0(\mathbf{K}(\mathbf{f}; M)) = (\ker(\partial_0) \otimes 1)/(\mathrm{im}(\partial_1) \otimes 1) \cong M/(\mathbf{f})M$. For homology groups of degree $i > 0$, we shall proceed by induction. If $r = 1$, we have the sequence

$$0 \xrightarrow{d_1} M \xrightarrow{f_1} M \longrightarrow 0,$$

where f_1 denotes multiplication by f_1, and thus

$$H_1(\mathbf{K}(\mathbf{f}; M)) = \ker(f_1)/\mathrm{im}(d_1) = \ker(f_1) = \{m \in M \mid f_1 m = 0\}.$$

As f_1 is a regular, $f_1 m = 0$ implies that $m = 0$, and thus $H_1(\mathbf{K}(\mathbf{f}; M)) = 0$.

Assume that the statement holds for a regular sequence $\mathbf{f}' = f_1, \ldots, f_{r-1}$. Let $\mathbf{K}' = \mathbf{K}(\mathbf{f}'; M)$. By the Lemma 5.10, we get the exact sequence

$$H_1(\mathbf{K}') \longrightarrow H_1(\mathbf{K}(\mathbf{f}; M)) \longrightarrow H_0(\mathbf{K}') \xrightarrow{f_q} H_0(\mathbf{K}').$$

By the induction assumption $H_1(\mathbf{K}') = 0$ and $H_0(\mathbf{K}') = M/\mathbf{f}'M$. As f_q is a regular element, $H_1(\mathbf{K}(\mathbf{f}; M)) = 0$. By Lemma 5.10, the sequence

$$H_i(\mathbf{K}') \longrightarrow H_i(\mathbf{K}(\mathbf{f}; M)) \longrightarrow H_{i-1}(\mathbf{K}')$$

is exact. By the induction assumption $H_i(\mathbf{K}') = H_{i-1}(\mathbf{K}') = 0$, and thus $H_i(\mathbf{K}(\mathbf{f}; M)) = 0$. □

The above theorem yields the main result of this section, which is the following.

Corollary 5.12. The Koszul complex of $x_1, \ldots, x_n$ in S is a minimal free resolution of $\Bbbk = S/\mathfrak{m}$.

Proof. Firstly, let us show that $x_1, \ldots, x_n$ is a regular sequence in S. Clearly, $(x_1, \ldots, x_n) \neq S$, as $a \notin (x_1, \ldots, x_n)S$ for any nonzero $a \in \Bbbk$. Next, let g be an element in the quotient $S/(x_1, \ldots, x_{k-1})$. Then $x_k g = 0$ implies that $g = 0$, as $x_k g \in (x_1, \ldots, x_{k-1})S$ if and only if $g \in (x_1, \ldots, x_{k-1})S$, as $(x_1, \ldots, x_{k-1})$ is a prime ideal.

As the sequence $x_1, \ldots, x_n$ is regular, $H_i(\mathbf{K}(\mathbf{f}; S)) = 0$ for $i > 0$ and $H_0(\mathbf{K}(\mathbf{f}; S)) = S/\mathfrak{m}$. From the definition of the boundary maps of $\mathbf{K}(\mathbf{f})$ we see that $\partial_i(K_i) \subseteq \mathfrak{m}K_{i-1}$, as none of the elements in our sequence are constants. Thus $\mathbf{K}(\mathbf{f}; S)$ is a minimal free resolution of $\Bbbk = S/\mathfrak{m}$. □

From now on we will denote $K_\circ = \mathbf{K}(x_1, \ldots, x_n)$ for the Koszul complex of $x_1, \ldots, x_n$ in the polynomial ring S.

When specifying the Koszul complex for our purposes, we can use monomial matrices to define the Koszul complex $K_\circ$ of $x_1, \ldots, x_n$. This turns out to be useful, as it gives us a clear relation between the minimal free resolution of $S/\mathfrak{m}$ and the reduced chain complex of the full simplex.

Let Δ be the full simplex consisting of all subsets of $[n]$. First, we construct the reduced chain complex of Δ, label the column and row corresponding to e_σ by $\mathbf{x}^\sigma$, and shift the homological degrees so that the module corresponding to the empty set is in degree 0. Now, an element $\mathbf{x}^\sigma$ is nonzero in the free module $S(-\tau)$, which is generated by 1_τ, exactly when $\mathbf{x}^\tau$ divides $\mathbf{x}^\sigma$.

Example 5.13. Consider the sequence x, y, z in the polynomial ring $S = \Bbbk[x, y, z]$. The Koszul complex of this sequence is given by

$$K_\circ \ : \quad 0 \longrightarrow S \xrightarrow{\begin{array}{cc} & xyz \\ \begin{array}{c} yz \\ xz \\ xy \end{array} & \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{array}} S^3 \xrightarrow{\begin{array}{cc} & \begin{array}{ccc} yz & xz & xy \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \end{array}} S^3 \xrightarrow{\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ 1 & \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{array}} S \longrightarrow 0.$$

If we disregard the labeling of the matrices, and replace the free modules S^m with vector spaces $\Bbbk^m$, we are left with the reduced chain complex of the full simplex on the vertex set [3]. Note that each free module has the same rank as K_j in the more general construction using exterior powers, namely $\binom{n}{j}$ for a sequence of n elements. Recall that every free R-module of rank k is isomorphic to R^k.

When using the full simplex on vertex set $[n]$, the above construction gives us the Koszul complex of $x_1, \ldots, x_n$ in the polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ [6, p. 13]. For any $\mathbf{a} \in \mathbb{N}^n$ with support σ, we can examine the degree-$\mathbf{a}$ part of $K_\circ$ by restricting to the rows and columns of the matrix labeled by faces of σ. Thus the restriction $(K_\circ)_\mathbf{a}$ to degree $\mathbf{a}$ is the reduced chain complex of σ when we consider complex of $\Bbbk$-vector spaces instead of a complex of free modules, disregarding the monomials in the boundary maps. All simplexes which we can restrict to are cones, with the exception of the face $\emptyset$, which is the only face that contributes to the degree $(0, \ldots, 0)$ part of $K_\circ$. By Lemma 4.14, it follows that the homology groups of $K_\circ$ are 0 in all positive homological degrees, so $K_\circ$ is exact, and $H_0 = k$. Thus $K_\circ$ is the minimal free resolution of $\Bbbk = S/\mathfrak{m}$.

Example 5.14. Let $K_\circ$ be the Koszul complex from Example 5.13, and $\mathbf{a} = (2, 3, 0)$. The support of $\mathbf{a}$ is $\sigma = \{1, 2\}$, so the degree-$\mathbf{a}$ part of $K_\circ$ is

$$(K_\circ)_\mathbf{a}\colon \quad 0 \longrightarrow S \xrightarrow{\begin{array}{cc} & xy \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}} S^2 \xrightarrow{\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ 1 & \begin{bmatrix} 1 & 1 \end{bmatrix} \end{array}} S \longrightarrow 0.$$

5.3 Subresolutions

In the next section, we will use properties of subresolutions to show the case of non-squarefree ideals in Hochster's formula. The general theory and proof of Proposition 5.16 can be found in [7, Chapter 56].

Definition 5.15. Let $\mathbf{F}$ be a free resolution of a monomial ideal I, and m be a monomial in I. The complex $\mathbf{F}(\leq m)$ is defined as the subcomplex of $\mathbf{F}$ with free modules generated by the homogenous basis elements with multidegrees that divide m.

Proposition 5.16. [7, Proposition 56.1.] Let $\mathbf{F}$ be a free resolution of S/I, where I is an ideal generated by the monomials $m_1, \dots, m_r$. Let $u \in I$ be a monomial, and $I_{\leq u}$ be the monomial ideal generated by momomials $\{m_i \mid m_i \text{ divides } u\}$. If $\mathbf{F}$ is a minimal free resolution of S/I, then $\mathbf{F}(\leq u)$ is a minimal free resolution of $S/I_{\leq u}$.

Example 5.17. Let $S = \Bbbk[x, y, z]$, $I = (x^2, xy, yz, y^2)$ and $m = xyz$. The minimal free resolution of I, using standard notation for the boundary maps, is

$$\mathbf{F}\colon\ 0 \longrightarrow S \xrightarrow{\begin{bmatrix} 0 \\ z \\ -y \\ x \end{bmatrix}} S^4 \xrightarrow{\begin{bmatrix} -y & 0 & 0 & 0 \\ x & -y & -z & 0 \\ 0 & x & 0 & -z \\ 0 & 0 & x & y \end{bmatrix}} S^4 \xrightarrow{\begin{bmatrix} x^2 & xy & y^2 & yz \end{bmatrix}} I \longrightarrow 0.$$

The ideal $I_{\leq m}$ is $I_{\leq m} = (xy, yz)$, and its minimal free resolution is given by the subresolution

$$\mathbf{F}_{(\leq m)}\colon\ 0 \longrightarrow S \xrightarrow{\begin{bmatrix} -z \\ x \end{bmatrix}} S^2 \xrightarrow{\begin{bmatrix} xy & yz \end{bmatrix}} I_{\leq m} \longrightarrow 0.$$

6 Hochster's formula

This section is dedicated to the proof of the main theorem of this thesis, Hochster's formula. Both the dual and original form of the formula will be proved in this section, and we will also show how it can be used to compute the Betti numbers of general monomial ideals using polarization.

Definition 6.1. The **upper Koszul simplicial complex** in degree $\mathbf{b} \in \mathbb{N}^n$ of a monomial ideal I is defined as

$$K^{\mathbf{b}}(I) = \{\tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I,\ \tau \text{ squarefree}\}.$$

We will use the upper Koszul simplicial complex to relate the Betti numbers of an ideal to the dimension of the homology groups of $K^{\mathbf{b}}(I)$. This will allow us to give the Betti numbers of squarefree monomial ideals in terms of the dimensions of the homology groups of the corresponding simplicial complex.

Proposition 6.2. Let Δ be a simplicial complex on $[n]$ and $\tau \subseteq [n]$.

(i) $\Delta^\vee = K^{\mathbf{1}}(I_\Delta)$

(ii) $K^\tau(I_\Delta) = \mathrm{link}_{K^{\mathbf{1}}(I_\Delta)}(\bar{\tau})$

Proof. (i) From the definitions of the Alexander dual (which consists of the complements of the non-faces of Δ) and the upper Koszul simplicial complex, we get

$$\begin{aligned} \sigma \in K^{\mathbf{1}}(I_\Delta) &\Leftrightarrow \mathbf{x}^{\mathbf{1}-\sigma} \in I_\Delta \\ &\Leftrightarrow \mathbf{1} - \sigma \notin \Delta \\ &\Leftrightarrow \sigma \in \Delta^\vee. \end{aligned}$$

(ii) Using the definition and the result above

$$\begin{aligned} \sigma \in K^{\tau}(I_\Delta) &\Leftrightarrow \mathbf{x}^{\tau-\sigma} \in I_\Delta \\ &\Leftrightarrow \tau - \sigma \notin \Delta,\ \sigma \subset \tau \\ &\Leftrightarrow \tau \cap \bar{\sigma} \notin \Delta,\ \sigma \subset \tau \\ &\Leftrightarrow \bar{\tau} \cup \sigma \in \Delta^\vee,\ \sigma \cap \bar{\tau} = \emptyset \\ &\Leftrightarrow \sigma \in \mathrm{link}_{\Delta^\vee}(\bar{\tau}) \\ &\Leftrightarrow \sigma \in \mathrm{link}_{K^{\mathbf{1}}(I_\Delta)}(\bar{\tau}). \end{aligned}$$

□

The following theorem relates the Betti numbers of an ideal to the dimension of a homology group of its upper Koszul simplicial complex. We will prove this using Lemma 5.3. As shown in Theorem 5.4, the modules $\mathrm{Tor}_i^S(M,N)$ can be found by tensoring a free resolution of M with N and calculating the homology groups, or alternatively constructing a free resolution of N and applying $\_ \otimes M$ to the resolution. This allows us to consider the minimal free resolution of $\Bbbk = S/\mathfrak{m}$, namely the Koszul complex $K_\circ$, when computing $\mathrm{Tor}_i^S(\Bbbk, I)_{\mathbf{a}}$ in the proof of the following theorem.

Theorem 6.3. The Betti numbers of a monomial ideal $I \subseteq S$ in degree $\mathbf{a} \in \mathbb{N}^n$ are given by

$$\beta_{i,\mathbf{a}}(I) = \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{a}}(I); \Bbbk).$$

Proof. By Lemma 5.3 and Theorem 5.4, we have

$$\beta_{i,\mathbf{a}} = \dim_k \mathrm{Tor}_i^S(I, \Bbbk)_{\mathbf{a}} = \dim_k \mathrm{Tor}_i^S(\Bbbk, I)_{\mathbf{a}}.$$

Consider the complex $K_\circ \bigotimes I$:

$$K_\circ \bigotimes I : 0 \longrightarrow K_n \bigotimes I \xrightarrow{d_n \otimes 1} \dots \xrightarrow{d_2 \otimes 1} K_1 \bigotimes I \xrightarrow{d_1 \otimes 1} K_0 \bigotimes I \longrightarrow 0.$$

This is a subcomplex of $K_\circ$, which follows from the fact that I is a submodule of S, and $K_\circ \bigotimes S = K_\circ$, so $K_\circ \bigotimes I \subseteq K_\circ \bigotimes S = K_\circ$. The complex $(K_\circ \bigotimes I)_{\mathbf{a}}$ is a subcomplex of $(K_\circ)_{\mathbf{a}}$. Using monomial matrices to describe the boundary maps, the degree-$\mathbf{a}$ part of $K_\circ$, considered as a complex of vector spaces, is the reduced chain complex of the simplex with face supp($\mathbf{a}$). We want to identify which faces of the simplex contribute to $\Bbbk$-basis vectors to $(K_\circ)_{\mathbf{a}}$, and what happens to the summands when tensored with I. A summand of the Koszul complex $K_\circ$ corresponding to a squarefree vector τ is the rank 1 free module $S(-\tau)$. Tensoring this summand with I yields $I(-\tau)$, which contributes a nonzero vector space to degree $\mathbf{a}$ if and only if $\mathbf{x}^{\mathbf{a}-\tau} \in I$.

The relationship between these complexes can be established more explicitly using the usual definition of the Koszul complex, with $K_i = \Lambda^i F$ being the i^{th} free module of the complex. Let $F = \{j_1 < \dots < j_i\} \subseteq [n]$, and $e_{j_1} \wedge \dots \wedge e_{j_i}$ be the basis elements of the free module K_i. Denote by $\epsilon(F)$ the squarefree vector in $\mathbb{N}^n$ with $\mathrm{supp}(\epsilon(F)) = F$. A $\Bbbk$-basis for the degree $\mathbf{a}$ component of $K_i \bigotimes I$ is given by elements

$$\mathbf{x}^{\mathbf{b}} \otimes (e_{j_1} \wedge \dots \wedge e_{j_i}),$$

such that $\mathbf{b} + \epsilon(F) = \mathbf{a}$ and $\mathbf{x}^{\mathbf{b}} \in I$. Thus we require that $\mathbf{x}^{\mathbf{a}-\epsilon(F)} \in I$, which gives us a one-to-one correspondence between these basis elements and the $(i-1)$-faces $\{j_1, \dots, j_i\}$ of the upper Koszul simplicial complex $K^{\mathbf{a}}(I)$, which determine the dimension of each vector space in the augmented chain complex. More precisely, the map of complexes is given by

$$\begin{aligned} \alpha :\ & \widetilde{\mathcal{C}_\circ}(K^{\mathbf{a}}(I); \Bbbk)[-1] \to ((K_\circ)_i \bigotimes I)_{\mathbf{a}} \\ \alpha_i :\ & F = \{j_1, \dots j_i\} \mapsto \ \mathbf{x}^{\mathbf{a}-\epsilon(F)} \otimes e_{j_1} \wedge \dots \wedge e_{j_i}, \end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Bbbk^{F_i(K^{\mathbf{a}}(I))} & \xrightarrow{\partial_i} & \Bbbk^{F_{i-1}(K^{\mathbf{a}}(I))} & \xrightarrow{\partial_{i-1}} & \Bbbk^{F_{i-2}(K^{\mathbf{a}}(I))} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i-1}} & & \downarrow{\scriptstyle \alpha_{i-2}} & & \\
\cdots & \longrightarrow & (K_{i+1} \otimes I)_{\mathbf{a}} & \xrightarrow{d_{i+1}\otimes 1} & (K_i \otimes I)_{\mathbf{a}} & \xrightarrow{d_i \otimes 1} & (K_{i-1} \otimes I)_{\mathbf{a}} & \longrightarrow & \cdots
\end{array}$$

As α is an isomorphism of complexes, it follows that $H_i(I \otimes K_\circ)_{\mathbf{a}} \cong \widetilde{H}_{i-1}(K^{\mathbf{a}}(I); \Bbbk)$, and thus $\beta_{i,\mathbf{a}}(I) = \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{a}}(I); \Bbbk)$. $\square$

Theorem 6.4 (**Hochster's formula, dual form**)**.** The nonzero Betti numbers of I_Δ and S/I_Δ are in squarefree degrees $\mathbf{a} \in \{0,1\}^n$, and are given by

$$\beta_{i,\mathbf{a}}(I_\Delta) = \beta_{i+1,\mathbf{a}}(S/I_\Delta) = \dim_{\Bbbk} \widetilde{H}_{i-1}(\mathrm{link}_{\Delta^\vee}(\bar{\mathbf{a}}); \Bbbk).$$

Proof. For squarefree $\mathbf{a}$, by Theorem 6.3 and Lemma 6.2 we have

$$\begin{aligned}
\beta_{i,\mathbf{a}}(I_\Delta) &= \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{a}}(I_\Delta); \Bbbk) \\
&= \dim_{\Bbbk} \widetilde{H}_{i-1}(\mathrm{link}_{K^{\mathbf{1}}(I_\Delta)}(\bar{\mathbf{a}}); \Bbbk) \\
&= \dim_{\Bbbk} \widetilde{H}_{i-1}(\mathrm{link}_{\Delta^\vee}(\bar{\mathbf{a}}); \Bbbk).
\end{aligned}$$

For non-squarefree degrees $\mathbf{a} \notin \{0,1\}^n$, consider the minimal free resolution $\mathbf{F}$ of S/I_Δ

$$\mathbf{F}: \quad 0 \longrightarrow F_\ell \xrightarrow{d_\ell} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} S/I_\Delta \longrightarrow 0.$$

Let $v = \prod_{i=1}^n x_i$. The subresolution $\mathbf{F}(\leq v)$ is given by

$$\mathbf{F}(\leq v): \quad 0 \longrightarrow F_\ell(\leq v) \xrightarrow{d_\ell} \cdots \xrightarrow{d_1} F_0(\leq v) \xrightarrow{\epsilon} S/I_\Delta(\leq v) \longrightarrow 0.$$

By Proposition 5.16, $\mathbf{F}(\leq v)$ is a minimal free resolution of $S/I_\Delta(\leq v)$. As $S/I_\Delta = S/I_\Delta(\leq v)$, the complex $\mathbf{F}(\leq v)$ is a minimal free resolution of S/I_Δ. The restriction $(\leq v)$ eliminates all components of $F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$ in degrees $\mathbf{a}$ which are not squarefree, so $\beta_{i,\mathbf{a}} = 0$ if $\mathbf{a} \notin \{0,1\}^n$.

The case of non-squarefree degrees can also be proved with simplicial methods. For degree $\mathbf{a}$ with $a_i \geq 2$ for some $i \in [n]$, $\mathbf{x}^{\mathbf{a}-(\tau \cup i)}$ is in I_Δ if and only if $\mathbf{x}^{\mathbf{a}-\tau}$ is. From the definition of the upper Koszul simplicial complex, we see that $K^{\mathbf{a}}(I_\Delta)$ is a cone with apex i, and thus by Proposition 4.14, $\widetilde{H}_i(\Delta; \Bbbk) = 0$ for all i. $\square$

Hochster's original proof of the formula was through cohomology. To present the original form, we apply Alexander duality and the following lemma to the dual form.

Lemma 6.5. Let Δ be a simplicial complex on $[n]$ and $\tau \subset [n]$. Then,

(i) $K^\tau(I_\Delta) = (\Delta|_\tau)^\vee$

(ii) $\mathrm{link}_{\Delta^\vee}(\bar{\tau}) = (\Delta|_\tau)^\vee$.

Proof. (i) Directly from the definitions we get

$$\begin{aligned}\sigma \in K^\tau(I_\Delta) &\Leftrightarrow \mathbf{x}^{\tau-\sigma} \in I_\Delta \\ &\Leftrightarrow \tau - \sigma \notin \Delta, \ \sigma \subset \tau \\ &\Leftrightarrow \sigma \in (\Delta|_\tau)^\vee.\end{aligned}$$

(ii) This follows directly from Proposition 6.2 and the above result, as

$$\mathrm{link}_{\Delta^\vee}(\bar{\tau}) = \mathrm{link}_{K^1(I_\Delta)}(\bar{\tau}) = K^\tau(I_\Delta) = (\Delta|_\tau)^\vee.$$

□

Theorem 6.6 (**Hochster's formula**). The nonzero Betti numbers of I_Δ and S/I_Δ are in squarefree degrees $\tau \in \{0,1\}^n$, and are given by

$$\beta_{i,\tau}(I_\Delta) = \beta_{i+1,\tau}(S/I_\Delta) = \dim_{\Bbbk} \widetilde{H}^{|\tau|-i-1}(\Delta|_\tau; \Bbbk).$$

Proof. Applying Theorem 4.12 and the above Lemma to the dual form in Theorem 6.4, we get

$$\begin{aligned}\beta_{i,\tau}(I_\Delta) &= \dim_{\Bbbk} \widetilde{H}_{i-1}(\mathrm{link}_{\Delta^\vee}(\bar{\tau}); \Bbbk) \\ &= \dim_{\Bbbk} \widetilde{H}_{i-1}((\Delta|_\tau)^\vee; \Bbbk) \\ &= \dim_{\Bbbk} \widetilde{H}^{|\tau|-2-i}(\Delta|_\tau; \Bbbk).\end{aligned}$$

□

Even though Hochster's formula is strictly for squarefree monomial ideals, it can be applied to general monomial ideals through *polarization*, which deforms a monomial ideal into a squarefree monomial ideal in a larger polynomial ring without changing the Betti numbers of the ideal.

Definition 6.7. Let $I \subset S = \Bbbk[x_1, \ldots, x_n]$ be a monomial ideal generated by the monomials $m_1, \ldots, m_r$, with $m_i = \prod_{j=1}^n x_j^{a_{ij}}$ for $i = 1, \ldots r$. For each j let $a_j = \max\{a_{ij} \mid i = 1, \ldots, r\}$. Let $\widetilde{S}$ be the polynomial ring over $\Bbbk$ in the variables

$$x_{11}, x_{12}, \ldots, x_{1a_1}, x_{21}, \ldots, x_{2a_2}, \ldots, x_{n1}, \ldots, x_{na_n}.$$

The **polarization** $\widetilde{I} \subset \widetilde{S}$ of I is the ideal generated by the squarefree monomials $u_1, \ldots, u_r$, with

$$u_i = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk} \quad \text{for} \quad i = 1, \ldots, r.$$

For example, let I be the ideal $(x^3y^3, xy^2z^2, x^2z^2) \subset S = \Bbbk[x,y,z]$. The polarization of I is the squarefree monomial ideal

$$\widetilde{I} = (x_1x_2x_3y_1y_2y_3, x_1y_1y_2z_1z_2, x_1x_2z_1z_2)$$

in the ring $\widetilde{S} = \Bbbk[x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2]$.

The original module S/I can be recovered by taking the quotient of $\widetilde{S}/\widetilde{I}$ modulo the differences

$$x_{11} - x_{12}, \dots, x_{11} - x_{1a_1}, \dots x_{21} - x_{22}, \dots, x_{n1} - x_{n2}, \dots, x_{n1} - x_{na_n},$$

as we want to set $x_{i,1} = \cdots = x_{i,a_i}$ for all $i = 1, \dots, n$. One of the key features of polarization is that these differences form an $\widetilde{S}/\widetilde{I}$-regular sequence, so that taking the above quotient preserves the homological invariants of the module. This fact can be used to prove the following.

Theorem 6.8. Let $\widetilde{I} \subset \widetilde{S}$ be the polarization of a monomial ideal $I \subset S$. Then

$$\beta_{i,j}(I) = \beta_{i,j}(\widetilde{I}).$$

See [4, p. 19] for a proof of this theorem.

This shows that the Betti numbers of general monomial ideals can be computed by polarizing the ideal and using Hochster's formula on the squarefree polarization. The drawback of polarization is that the simplicial complex corresponding to the polarized ideal is often much too large, and thus this technique is rarely used in practice for computing the Betti numbers.

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