DISCRETE MIXED VOLUMES AND ALEXANDROV–FENCHEL-TYPE INEQUALITIES

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Abstract. Motivated by the parallels between volume and lattice point counting, we study the mixed version of Ehrhart theory, computing the number of lattice points in Minkowski sums of lattice polytopes. This quantity is a polynomial in the scalars, and the coefficients in the binomial basis are given by the discrete mixed volumes, which generalize mixed volumes in the case of lattice polytopes. We prove a general asymptotic Alexandrov–Fenchel-type inequality for the discrete mixed volumes of lattice polytopes, and an exact inequality for coordinate simplices. To prove the exact inequality, we use the exceptional Hirzebruch–Riemann–Roch theorem of Berget–Eur–Spink–Tseng on the permutohedral variety to express the discrete mixed volume as a classical mixed volume. Additionally, we give a matroidal interpretation of this result, showing that these quantities are exactly the coefficients of the characteristic polynomial of the associated cotransversal matroid, recovering their log-concavity.

1. Introduction

For a lattice d-polytope $P \subset \mathbb{R}^d$, Ehrhart [Ehr62] proved that the lattice-point enumerator in integer multiples of P is a polynomial of degree d. We denote this polynomial by

$$E_P(t) := |tP \cap \mathbb{Z}^d| = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0.$$

The leading coefficient is $a_d = \mathrm{vol}(P)$, the constant term is $a_0 = 1$, and $a_{d-1} = \frac{1}{2}\sum_F \mathrm{vol}_{\mathbb{Z}}(F)$, where the sum is over the facets F of P, and $\mathrm{vol}_{\mathbb{Z}}(F)$ denotes the $(d-1)$-dimensional volume normalized with respect to the lattice induced on the affine span of F. The remaining coefficients $a_{d-2}, \ldots, a_1$ may be negative in general [HHTY19]. However, Stanley proved that the h^*-polynomial, which is the numerator of the rational expression for the Hilbert series of the associated semigroup algebra, is monotone with respect to containment of lattice polytopes, and hence nonnegative [Sta93].

Turning to the mixed situation, let $P_1, \ldots, P_k \subseteq \mathbb{R}^d$ be convex bodies. The volume $\mathrm{vol}(n_1 P_1 + \cdots + n_k P_k)$ is a homogeneous polynomial of total degree d.

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The mixed volumes are characterized by

$$\mathrm{vol}(n_1P_1+\cdots+n_kP_k)=\sum_{\substack{\alpha\in\mathbb{Z}^k_{\geq 0}\\ |\alpha|=d}}\binom{d}{\alpha}\mathrm{MV}(P_1[\alpha_1],\ldots,P_k[\alpha_k])\,\mathbf{n}^\alpha,$$

where $P_i[\alpha_i]$ denotes α_i-copies of P_i. In particular, for d convex bodies $P_1,\ldots,P_d$, the coefficient of $n_1\cdots n_d$ is

$$d!\,\mathrm{MV}(P_1,\ldots,P_d).$$

Mixed volumes satisfy many remarkable properties, including multilinearity, monotonicity with respect to containment (and hence nonnegativity), and the Alexandrov–Fenchel inequalities

$$\mathrm{MV}(P_1,\ldots,P_d)^2\geq \mathrm{MV}(P_1,P_1,P_3,\ldots,P_d)\mathrm{MV}(P_2,P_2,P_3,\ldots,P_d), \tag{1}$$

corollaries of which include the Brunn-Minkowski inequality and the isoperimetric inequality for convex bodies. The Alexandrov–Fenchel and the Khovanskii-Teissier inequalities in intersection theory, related through the volume polynomial on toric varieties, have also seen many applications in combinatorics, starting in [Sta81].

For lattice polytopes, a natural generalization of Ehrhart theory is to consider the multivariate lattice-point enumerator $\left|(n_1P_1+\cdots+n_kP_k)\cap\mathbb{Z}^d\right|$. This is again a polynomial of degree $d_i=\dim P_i$ in n_i, though not homogeneous. The coefficients of this polynomial in the binomial basis are given by the *discrete mixed volumes* (DMVs) of subcollections of $(P_1,\ldots,P_k)$. These are a special case of the *combinatorial mixed valuations* associated to translation-invariant valuations studied by Jochemko–Sanyal [JS17].

In this paper, we study Alexandrov–Fenchel-type inequalities for the discrete mixed volumes of collections of lattice polytopes. In Theorem 3.4, we prove a general asymptotic version of these inequalities, and in Theorem 4.4 an exact version for coordinate simplices, where the proof relies on an exceptional version of the Hirzebruch–Riemann–Roch theorem on the permutohedral variety, proved by Berget–Eur–Spink–Tseng [BEST23]. This exact version of the inequality follows from an expression of the discrete mixed volume of coordinate simplices as a classical mixed volume of the collection, padded with the negative coordinate simplex. A collection of coordinate simplices corresponds to a set system, and hence a (co)transversal matroid. We prove that the coefficients of the characteristic polynomial of this cotransversal matroid are given by the discrete mixed volume of these simplices, simultaneously giving a proof of their log-concavity.

We also demonstrate that such AF-type inequalities do not hold in general; see Example 2.4.

2. Discrete mixed volumes

Definition 2.1. *Let $\mathcal{P} := (P_1, \ldots, P_k) \subset \mathbb{R}^d$ be lattice polytopes with $k \geq 0$. The* discrete mixed volume (DMV) *of $\mathcal{P}$ is given by*

$$\mathrm{DMV}(P_1, \ldots, P_k) := \sum_{I \subseteq [k]} (-1)^{k-|I|} \, \big|(\sum_{i \in I} P_i) \cap \mathbb{Z}^d\big|, \tag{2}$$

where $\sum_{i \in \varnothing} P_i := \{0\}$. In particular, $\mathrm{DMV}(\emptyset) = 1$.

For $k = 2$, $\mathrm{DMV}(P, Q) = |(P+Q) \cap \mathbb{Z}^d| - |P \cap \mathbb{Z}^d| - |Q \cap \mathbb{Z}^d| + 1$.. The fact that the discrete mixed volume is a translation-invariant valuation in each argument follows from the lattice point enumerator having this property.

Let

$$\mathcal{P} = (P_1, \ldots, P_k)$$

be a fixed collection of lattice polytopes, and define its multivariate Ehrhart polynomial by

$$E_{\mathcal{P}}(n_1, \ldots, n_k) := \left|(n_1 P_1 + \cdots + n_k P_k) \cap \mathbb{Z}^d\right|.$$

For a function $f : \mathbb{Z}^k \to \mathbb{Z}$, define

$$\Delta_i f := f(n_1, \ldots, n_i + 1, \ldots, n_k) - f(n_1, \ldots, n_i, \ldots, n_k)$$

to be the finite difference operator in the i-th variable. For a multi-index $\boldsymbol{\alpha} \in \mathbb{Z}^k_{\geq 0}$, let

$$\Delta^{\boldsymbol{\alpha}} f := \Delta_1^{\alpha_1} \cdots \Delta_k^{\alpha_k} f.$$

As for all multivariate numerical polynomials, we may expand $\mathrm{E}_{\mathcal{P}}$ in the binomial basis as

$$\mathrm{E}_{\mathcal{P}}(\mathbf{n}) = \sum_{\alpha \in \mathbb{Z}^k_{\geq 0}} \Delta^{\boldsymbol{\alpha}} \mathrm{E}_{\mathcal{P}}(\mathbf{0}) \binom{\mathbf{n}}{\boldsymbol{\alpha}},$$

where $\binom{\mathbf{n}}{\boldsymbol{\alpha}} := \binom{n_1}{\alpha_1} \cdots \binom{n_k}{\alpha_k}$. Thus, the coefficient of $\binom{\mathbf{n}}{\boldsymbol{\alpha}}$ is exactly the discrete mixed volume of the collection of lattice polytopes indexed by $\boldsymbol{\alpha}$: $\Delta^{\boldsymbol{\alpha}} \mathrm{E}_{\mathcal{P}}(\mathbf{0}) = \mathrm{DMV}(P_1[\alpha_1], \ldots, P_k[\alpha_k])$, where $P_i[\alpha_i]$ denotes α_i copies of P_i.

As in the case of univariate Ehrhart polynomials, the number of lattice points in $\sum_i n_i P_i$ is asymptotically equal to $\mathrm{vol}(\sum_i n_i P_i)$, so for the top degree coefficients $|\alpha| = \sum_i \alpha_i = d$, we have

$$\mathrm{DMV}(P_1[\alpha_1], \ldots, P_k[\alpha_k]) = d! \mathrm{MV}(P_1[\alpha_1], \ldots, P_k[\alpha_k]). \tag{3}$$

This expression for the mixed volumes of lattice polytopes was first proved by Bernstein [Ber76]. Thus, the leading coefficients of mixed Ehrhart polynomials satisfy the Alexandrov–Fenchel inequalities.

For the lower-degree coefficients of a mixed Ehrhart polynomial, a fundamental general result is nonnegativity in the binomial basis, which follows from the monotonicity of discrete mixed volumes, proved by Jochemko and Sanyal.

Theorem 2.2 ([JS17, Corollary 1.2])**.** *If $(P_1, \ldots, P_k)$ and $(Q_1, \ldots, Q_k)$ are collections of lattice polytopes with $P_i \subseteq Q_i$ for all $1 \leq i \leq k$, then*

$$\mathrm{DMV}(P_1, \ldots, P_k) \leq \mathrm{DMV}(Q_1, \ldots, Q_k).$$

In particular, the discrete mixed volumes are nonnegative.

Furthermore, we have $\mathrm{DMV}(P_1, \ldots, P_k) > 0$ if and only if there exist linearly independent segments $S_i \subseteq P_i$ with vertices in $\mathbb{Z}^d$ [JS17, Theorem 3.17]. The nonnegativity of discrete mixed volumes for $k < d$ was first proved by Bihan [Bih16]. For an interpretation of discrete mixed volumes as the motivic arithmetic genus of a complete intersection associated to a collection of lattice polytopes, see [DRHN19].

The Alexandrov–Fenchel-type inequality we will investigate is the following:

Question 2.3. *For which collections of lattice polytopes $P, Q, K_3, \ldots, K_r \subset \mathbb{R}^d$ with $r \leq d$ do we have*

$$\mathrm{DMV}(P, Q, \mathcal{K})^2 \;\geq\; \mathrm{DMV}(P, P, \mathcal{K})\; \mathrm{DMV}(Q, Q, \mathcal{K}), \tag{4}$$

where $\mathcal{K} := (K_3, \ldots, K_r)$?

When $r = d$, this reduces to the Alexandrov-Fenchel inequalities by Equation (3). Whenever $\dim\left(P + Q + \sum_{i=3}^{r} K_i\right) = r$, the inequality is again equivalent to Alexandrov–Fenchel. Indeed, choose lattice vertices in $P, Q, K_3, \ldots, K_r$ and translate the polytopes by these lattice vectors so that each lies in the common linear span $L := \mathrm{span}\left(P + Q + \sum_{i=3}^{r} K_i - v\right)$, where v is any lattice point in the Minkowski sum. Since discrete mixed volumes are invariant under translating individual inputs by lattice vectors, the computation may be performed in the rank-r lattice $L \cap \mathbb{Z}^d$. With respect to this induced lattice, the top-degree formula above identifies the DMV inequality with the ordinary Alexandrov–Fenchel inequality in dimension r.

For the general case, we prove an asymptotic form under dilation of the first two inputs and an exact form for coordinate simplices. For general families of lattice polytopes, the inequality Equation (4) does not hold, as the following example shows.

Example 2.4. *Let $R_m = \mathrm{conv}\{(0,0,0), (1,0,0), (0,1,0), (1,1,m)\} \subset \mathbb{R}^3$ be the Reeve tetrahedron with height parameter m. The Ehrhart polynomial of R_m is given by*

$$E_{R_m}(t) = \frac{m}{6}t^3 + t^2 + (2 - \frac{m}{6})t + 1,$$

so that $E_{R_m}(2) = m + 9$, and $\mathrm{DMV}(R_m, R_m) = E_{R_m}(2) - 2E_{R_m}(1) + 1 = m + 2$. Let $P = R_m \times \{0\}^3 \subseteq \mathbb{R}^6$ and $Q = \{0\}^3 \times R_m \subseteq \mathbb{R}^6$ be orthogonal copies of R_m in $\mathbb{R}^6$. Then $|(P + Q) \cap \mathbb{Z}^6| = |(R_m \times R_m) \cap \mathbb{Z}^6| = 16$, and $\mathrm{DMV}(P, Q) = 9$. Thus $\mathrm{DMV}(P, Q)^2 = 9^2 < \mathrm{DMV}(P, P)\mathrm{DMV}(Q, Q) = (m + 2)^2$ for $m > 7$.

3. Asymptotic AF-type inequality

Fix a collection of lattice polytopes $\mathcal{P} = (P, Q, K_3, \dots, K_r) \subset \mathbb{R}^d$ with $2 \le r \le d$, and define for $n \in \mathbb{Z}_{\ge 0}$

$$\begin{aligned} A(n) &:= \mathrm{DMV}(nP, nQ, K_3, \dots, K_r),\\ B(n) &:= \mathrm{DMV}(nP, nP, K_3, \dots, K_r),\\ C(n) &:= \mathrm{DMV}(nQ, nQ, K_3, \dots, K_r). \end{aligned}$$

We will see that the asymptotic behaviour of these polynomials is determined by the mixed volumes of lattice polytopes in question. We will need the mixed version of the Brunn-Minkowski inequality, which can be derived from the Alexandrov–Fenchel inequalities (see, for example, [Sch14, Ch. 7]).

Theorem 3.1 (Mixed Brunn–Minkowski)**.** *Fix convex bodies $K_{m+1}, \dots, K_d$ in $\mathbb{R}^d$ for $1 \le m \le d$, interpreting the list as empty if $m = d$. Define, for a convex body L,*

$$G(L) := \mathrm{MV}(L[m], K_{m+1}, \dots, K_d),$$

where $L[m]$ denotes m copies of L. Then $G(L)^{1/m}$ is concave on the cone of convex bodies with respect to Minkowski addition and scaling. In particular, we have

$$G(L+R)^{1/m} \ \ge\ G(L)^{1/m} + G(R)^{1/m}.$$

Lemma 3.2. *Let $m := d - (r-2)$, and specialize the function G to $G(L) := \mathrm{MV}(L[m], K_3, \dots, K_r)$. Then, as $n \to \infty$, we have*

$$\begin{aligned} A(n) &= \frac{d!}{m!}\left(G(P+Q) - G(P) - G(Q)\right) n^m + O(n^{m-1}),\\ B(n) &= \frac{d!}{m!}\big(G(2P) - 2G(P)\big)\, n^m + O(n^{m-1}),\\ C(n) &= \frac{d!}{m!}\big(G(2Q) - 2G(Q)\big)\, n^m + O(n^{m-1}). \end{aligned}$$

Proof. Consider the multivariate Ehrhart polynomial

$$\mathrm{E}_{\mathcal{P}}(\mathbf{u}) := \big|(u_1P + u_2Q + u_3K_3 + \cdots + u_rK_r) \cap \mathbb{Z}^d\big|.$$

The top-degree homogeneous component is the volume polynomial

$$\sum_{\substack{\boldsymbol{\alpha} \in \mathbb{Z}^r_{\ge 0}\\ |\boldsymbol{\alpha}| = d}} \binom{d}{\alpha_1, \dots, \alpha_r} \mathrm{MV}(P[\alpha_1], Q[\alpha_2], K_3[\alpha_3], \dots, K_r[\alpha_r])\mathbf{u}^{\boldsymbol{\alpha}}$$

By definition, $A(n)$ is given by the finite difference

$$A(n) = \Delta^{(n)}_{u_1}\Delta^{(n)}_{u_2}\Delta_{u_3}\dots\Delta_{u_r}\mathrm{E}_{\mathcal{P}}(0),$$

where $\Delta^{(n)} f(k) = f(k+n) - f(k)$. For a monomial $u_1^{\alpha_1} u_2^{\alpha_2} \prod_{i=3}^r u_i^{\alpha_i}$, we have

$$\Delta_{u_1}^{(n)} \Delta_{u_2}^{(n)} \Delta_{u_3} \dots \Delta_{u_r} (u_1^{\alpha_1} u_2^{\alpha_2} \prod_{i=3}^r u_i^{\alpha_i}) = \begin{cases} n^{\alpha_1+\alpha_2} & \text{if } \alpha_i > 0 \ \forall i \\ 0 & \text{else.} \end{cases}$$

From this we see that $A(n)$ is a polynomial of degree at most $m := d - (r-2)$, and the coefficient of n^m in $A(n)$, which we denote by LC_A, is the sum of the coefficients of the monomials $\mathbf{u}^\alpha$ in $\mathrm{E}_{\mathcal{P}}$ with $\alpha_3 = \ldots = \alpha_r = 1$ and $\alpha_1 + \alpha_2 = m$, and hence $|\alpha| = d$ for these terms. Extracting these terms from the volume polynomial, we see that the LC_A is given by

$$\begin{aligned} \mathrm{LC}_A &= \sum_{a=1}^{m-1} \frac{d!}{a!(m-a)!} \mathrm{MV}(P[a], Q[m-a], K_3, \dots, K_r) \\ &= \frac{d!}{m!} (G(P+Q) - G(P) - G(Q)), \end{aligned}$$

where we have expanded $G(P+Q)$ using multilinearity of mixed volumes. The expressions for the leading coefficients for $B(n), C(n)$ follow similarly. □

Proposition 3.3. *Let $m = d - (r-2)$ as before. Then*

$$(G(P+Q) - G(P) - G(Q))^2 \ \geq \ (2^m - 2)^2 \, G(P) \, G(Q), \tag{5}$$

so the leading coefficients in Lemma 3.2 satisfy

$$\mathrm{LC}_A^2 \geq \ \mathrm{LC}_B \mathrm{LC}_C.$$

Proof. Write $x := G(P)^{1/m}$ and $y := G(Q)^{1/m}$. By Theorem 3.1, $G(P+Q)^{1/m} \geq x + y$, so $G(P+Q) \geq (x+y)^m$, and

$$G(P+Q) - G(P) - G(Q) \ \geq \ (x+y)^m - x^m - y^m = \sum_{i=1}^{m-1} \binom{m}{i} x^i y^{m-i}.$$

Using $\binom{m}{i} = \binom{m}{m-i}$ and the AM–GM inequality $x^i y^{m-i} + x^{m-i} y^i \geq 2 x^{m/2} y^{m/2}$,

$$\sum_{i=1}^{m-1} \binom{m}{i} x^i y^{m-i} = \frac{1}{2} \sum_{i=1}^{m-1} \binom{m}{i} \left(x^i y^{m-i} + x^{m-i} y^i \right) \ \geq \ \frac{1}{2} \sum_{i=1}^{m-1} \binom{m}{i} \cdot 2 \, x^{m/2} y^{m/2}.$$

Since $\sum_{i=1}^{m-1} \binom{m}{i} = 2^m - 2$, we obtain

$$G(P+Q) - G(P) - G(Q) \ \geq \ (2^m - 2) \, x^{m/2} y^{m/2} = (2^m - 2) \left(G(P) G(Q) \right)^{1/2},$$

which implies (5). If $m > 2$, equality in the AM–GM step occurs precisely when either $xy = 0$ or $x = y$. Hence the inequality between the leading coefficients is strict unless equality holds in the mixed Brunn–Minkowski inequality $G(P+Q)^{1/m} \ \geq \ G(P)^{1/m} + G(Q)^{1/m}$ and either $G(P) = G(Q)$ or $G(P)G(Q) = 0$. When $m = 2$, the AM–GM step is an identity. □

When $r = d$, the desired inequality follows directly from Equation (3) and Alexandrov–Fenchel. If $r < d$, then $m > 2$, and Proposition 3.3 gives the following result.

Theorem 3.4. *For collections of lattice polytopes* $(P, Q, \mathcal{K}) = (P, Q, K_3, \ldots, K_r) \subset \mathbb{R}^d$ *with* $r < d$, *we have*

$$\mathrm{DMV}(nP, nQ, \mathcal{K})^2 > \mathrm{DMV}(nP, nP, \mathcal{K})\ \mathrm{DMV}(nQ, nQ, \mathcal{K})$$

for $n \gg 0$, *with the possible exception of* $(P, Q, \mathcal{K})$ *that achieve both equality in the mixed Brunn-Minkowski inequality as well as* $G(P) = G(Q)$ *or* $G(P)G(Q) = 0$.

A similar asymptotic result on discrete mixed volumes appears in [HJKST17, Corollary 4.6.], where the authors prove that the *mixed* h^**-polynomial* $h^*(x) = h^*_d(\mathcal{P})x^d + \ldots + h^*_1(\mathcal{P})x + h^*_0(\mathcal{P})$ of a collection $\mathcal{P} = (P_1, \ldots, P_k)$, defined by the identity $\mathrm{DMV}(nP_1, \ldots, nP_k) = \sum_{i=0}^d h^*_i(\mathcal{P})\binom{n+d-i}{d}$, is real-rooted for a sufficiently large common dilate $(nP_1, \ldots, nP_k)$ when all the P_i are d-dimensional.

4. Discrete mixed volumes of coordinate simplices

4.1. **Toric geometry.** In this section we prove an exact AF inequality for coordinate simplices using the exceptional Hirzebruch–Riemann–Roch theorem of Berget–Eur–Spink–Tseng and the Khovanskii–Teissier inequality. To begin with, we give reminders of the basic notions in toric geometry required in this section, and refer to [CLS11] for details.

Let $\Sigma \subseteq N_{\mathbb{R}}$ be a complete smooth rational fan in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ for a lattice $N \cong \mathbb{Z}^d$, and let X_Σ be the associated toric variety. Define $K(\Sigma)$ to be the set of all lattice polytopes P in $M_{\mathbb{R}} := \mathrm{Hom}(N, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ whose normal fan Σ_P coarsens Σ. The normal fan Σ_{P+Q} of the Minkowski sum of polytopes P and Q is the coarsest common refinement of Σ_P and Σ_Q, so $K(\Sigma)$ is closed under Minkowski addition. We denote the Chow ring and K-ring of a variety X by $A^*(X)$ and $K(X)$, respectively.

Every $P \in K(\Sigma)$ defines a basepoint-free torus-invariant Cartier divisor D_P on X_Σ, a corresponding class $[D_P] \in A^1(X)$, and a line bundle $\mathcal{O}_{X_\Sigma}(D_P) \in K(X)$. Furthermore, we have

$$\mathcal{O}_{X_\Sigma}(D_{P+Q}) = \mathcal{O}_{X_\Sigma}(D_P + D_Q) = \mathcal{O}_{X_\Sigma}(D_P) \bigotimes \mathcal{O}_{X_\Sigma}(D_Q),$$

and

$$\chi(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_P)) = \dim H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D_P)) = |P \cap \mathbb{Z}^d|.$$

Thus

$$|\big(\sum_{i=1}^k n_i P_i\big) \cap \mathbb{Z}^d| \ = \chi(X_\Sigma, \bigotimes_{i=1}^k L_i^{n_i}),$$

where $L_i := \mathcal{O}_{X_\Sigma}(D_{P_i})$.

For a general projective variety X and line bundles $L_1, \ldots, L_k$ on X, the function $(n_1, \ldots, n_k) \mapsto \chi(X, \bigotimes_{i=1}^k L_i^{n_i})$ is known as the *Snapper polynomial* of the collection $\{L_i\}_{i=1}^k$. With this point of view on multivariate Ehrhart polynomials, we can express the discrete mixed volumes as follows.

Lemma 4.1. *Let Σ be a complete rational fan, $P_1, \dots, P_k \in K(\Sigma)$ lattice polytopes whose normal fans coarsen Σ, and $L_i = \mathcal{O}_X(D_{P_i})$ the corresponding line bundles on X_Σ. Then*

$$\mathrm{DMV}(P_1, \dots, P_k) = \chi(X_\Sigma, \prod_{i=1}^k ([L_i] - 1)).$$

Proof. This follows immediately from expanding $\bigotimes_{i=1}^k ([L_i] - 1)$ as a K-class, the fact that $\chi(X_\Sigma, L_{i_1} \otimes \cdots \otimes L_{i_r}) = |(\sum_{j=1}^r P_{i_j}) \cap \mathbb{Z}^d|$, and the definition of discrete mixed volumes. ☐

Thus, we want to compute Euler characteristics of products of K-classes of the form $[L] - 1$, where $L = \mathcal{O}_X(D_P)$ for a polytope P. By the Hirzebruch-Riemann-Roch theorem, we can compute the discrete mixed volume as

$$\mathrm{DMV}(P_1, \dots, P_k) = \int_{X_\Sigma} \mathrm{ch}\left(\prod_{i=1}^k \big([\mathcal{O}_{X_\Sigma}(D_{P_i})] - 1\big)\right) \mathrm{Td}(X_\Sigma).$$

where $\mathrm{ch}(L)$ is the Chern character of a line bundle L and $\mathrm{Td}(X)$ is the Todd class of X. The presence of the Todd class makes it challenging to use this for lattice-point counting in practice. However, for certain varieties especially relevant for matroid theory, exceptional versions of the Hirzebruch-Riemann-Roch theorem have been discovered in recent work [BEST23, EHL23, EFLS24, LLPP24], where the situation simplifies considerably. To compute discrete mixed volumes of simplices, we will focus here on the permutohedral variety, where an exceptional Hirzebruch-Riemann-Roch theorem is due to [BEST23].

4.2. **Permutohedral variety.** Fix $E := \{0, 1, \dots, d\}, \mathbf{1} := \sum_{i \in E} e_i \in \mathbb{Z}^E$. Let $N_E := \mathbb{Z}^E / \mathbb{Z}\mathbf{1}$ and $M_E := \mathrm{Hom}(N_E, \mathbb{Z}) = \left\{m \in \mathbb{Z}^E : \sum_{i \in E} m_i = 0\right\}$. Both lattices have rank d. For $F \subseteq E$, put

$$e_F := \sum_{i \in F} e_i,$$

and denote its image in N_E by $\overline{e}_F$.

For every chain of nonempty proper subsets $\mathcal{F} : \quad \varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_\ell \subsetneq E$, let

$$\sigma_{\mathcal{F}} := \mathrm{cone}\big(\overline{e}_{F_1}, \dots, \overline{e}_{F_\ell}\big) \subseteq N_{E,\mathbb{R}}.$$

The collection of these cones and their faces is the braid fan Σ_E. Equivalently, Σ_E is the normal fan of the permutohedron

$$\Pi(E) = \mathrm{conv}\left\{\big(\sigma(0), \dots, \sigma(d)\big) : \sigma \in \mathfrak{S}_E\right\} \subseteq \mathbb{R}^E,$$

modulo its lineality space $\mathbb{R}\mathbf{1}$. The fan Σ_E is a complete smooth fan of dimension d, and the permutohedral variety $X_E := X_{\Sigma_E}$ is the associated toric variety.

For a nonempty subset $F \subseteq E$, let

$$\widehat{\Delta}_F := \operatorname{conv}\{e_i : i \in F\} \subseteq \mathbb{R}^E.$$

This simplex lies in the affine hyperplane

$$H_1 := \left\{ x \in \mathbb{R}^E : \sum_{i \in E} x_i = 1 \right\}.$$

To regard it as a lattice polytope in $M_{E,\mathbb{R}}$, we use the translated representative

$$\Delta_F := \widehat{\Delta}_F - e_0 = \operatorname{conv}\{e_i - e_0 : i \in F\} \subseteq M_{E,\mathbb{R}}.$$

Changing the distinguished element 0 changes Δ_F only by a lattice translation, and hence does not affect lattice-point counting. All lattice-point counts are taken with respect to the lattice M_E. Volumes are normalized so that a fundamental parallelepiped of M_E has volume 1, and MV denotes the corresponding standard mixed volume, normalized by $\mathrm{MV}(P[d]) = \mathrm{vol}(P)$.

The support function of Δ_F is linear on every cone of the braid fan, and thus determines a nef toric divisor D_{Δ_F} on X_E. For every nonempty $F \subseteq E$, define

$$L_F := \mathcal{O}_{X_E}(D_{\Delta_F}), \qquad h_F := c_1(L_F) \in A^1(X_E), \qquad \eta_F := 1 - [L_F^{-1}] \in K(X_E).$$

Following the notation of [AHK18], we denote $\alpha := h_E$, which is the nef divisor class associated to the full standard simplex Δ_E, and let

$$\nabla_E := -\Delta_E, \qquad L_\nabla := \mathcal{O}_{X_E}(D_{\nabla_E}), \qquad \beta := c_1(L_\nabla).$$

Thus β is the nef divisor class associated to the negative standard simplex.

The following exceptional Hirzebruch–Riemann–Roch theorem, together with its description on the simplicial generators, follows from [BEST23, Theorem D and Section 10.4].

Theorem 4.2. *There is a ring isomorphism $\zeta : K(X_E) \to A^\bullet(X_E)$ characterized by the property that $\zeta(\eta_S) = h_S$, for which we have*

$$\chi(X_E, \xi) = \int_{X_E} \zeta(\xi) \cdot \frac{1}{1-\alpha}$$

for any $\xi \in K(X_E)$. Here, $\frac{1}{1-\alpha} = 1 + \alpha + \ldots + \alpha^d$.

As we want to compute Euler characteristics of products of K-classes of the form $[L_S - 1]$, we precompose with the duality involution $D : [\xi] \mapsto [\xi^\vee]$ of $K(X_E)$, so that $D([L_S - 1]) = [L_S^{-1} - 1] = -\eta_S$. Now, define $\widetilde{\zeta} := \zeta \circ D$, so that $\widetilde{\zeta}([L_S - 1]) = -h_S$. Since both D and ζ are ring isomorphisms, so is $\widetilde{\zeta}$. This isomorphism gives the following form of the exceptional Hirzebruch-Riemann-Roch theorem.

Proposition 4.3. *For every $\xi \in K(X_E)$,*

$$\chi(X_E, \xi) = (-1)^d \int_{X_E} \widetilde{\zeta}(\xi) \cdot \frac{1}{1+\beta}.$$

Proof. Let Z_F denote the torus-invariant prime divisor corresponding to the ray $\mathbb{R}_{\geq 0}\overline{e}_F$, for $\varnothing \subsetneq F \subsetneq E$. The canonical divisor of the smooth toric variety X_E is

$$K_{X_E} = -\sum_{\varnothing \subsetneq F \subsetneq E} Z_F.$$

Fixing $0 \in E$, the standard descriptions of the two distinguished nef classes give

$$\alpha = \sum_{\substack{\varnothing \subsetneq F \subsetneq E \\ 0 \in F}} [Z_F], \qquad \beta = \sum_{\substack{\varnothing \subsetneq F \subsetneq E \\ 0 \notin F}} [Z_F].$$

Hence

$$K_{X_E} = -(\alpha + \beta), \qquad \omega_{X_E} \simeq L_E^{-1} \otimes L_\nabla^{-1}.$$

By the defining property of ζ,

$$\zeta([L_E^{-1}]) = 1 - \alpha, \qquad \zeta([L_E]) = \frac{1}{1-\alpha}.$$

Furthermore, the proof of [BEST23, Theorem 10.11] gives

$$\zeta([L_\nabla]) = 1 + \beta.$$

Therefore

$$\zeta([\omega_{X_E}]) = \zeta([L_E^{-1}])\zeta([L_\nabla^{-1}]) = \frac{1-\alpha}{1+\beta}.$$

By Serre duality and Theorem 4.2, we have

$$\begin{aligned}
\chi(X_E, \xi) &= (-1)^d \chi\big(X_E, \omega_{X_E} \otimes D(\xi)\big) \\
&= (-1)^d \int_{X_E} \widetilde{\zeta}(\xi) \cdot \zeta([\omega_{X_E}]) \cdot \frac{1}{1-\alpha} \\
&= (-1)^d \int_{X_E} \widetilde{\zeta}(\xi) \cdot \frac{1-\alpha}{1+\beta} \cdot \frac{1}{1-\alpha} \\
&= (-1)^d \int_{X_E} \widetilde{\zeta}(\xi) \cdot \frac{1}{1+\beta}.
\end{aligned}$$

□

Theorem 4.4. *Let $I_1, \ldots, I_k \subseteq E$ be nonempty subsets, with $1 \leq k \leq d$. Then*

$$\mathrm{DMV}(\Delta_{I_1}, \ldots, \Delta_{I_k}) = d!\, \mathrm{MV}\big(\Delta_{I_1}, \ldots, \Delta_{I_k}, \nabla_E[d-k]\big). \tag{6}$$

In particular, the discrete mixed volumes of coordinate simplices satisfy the Alexandrov–Fenchel inequalities.

Proof. Set

$$\xi := \prod_{j=1}^{k}([L_{I_j}] - 1) \in K(X_E).$$

By the K-theoretic expression for discrete mixed volume and Proposition 4.3,

$$\begin{aligned}\mathrm{DMV}(\Delta_{I_1}, \dots, \Delta_{I_k}) &= \chi(X_E, \xi) \\ &= (-1)^d \int_{X_E} \widetilde{\zeta}(\xi) \cdot \frac{1}{1+\beta}.\end{aligned}$$

Since

$$\widetilde{\zeta}([L_{I_j}] - 1) = -h_{I_j},$$

we have

$$\widetilde{\zeta}(\xi) = (-1)^k h_{I_1} \cdots h_{I_k}.$$

Moreover,

$$\frac{1}{1+\beta} = \sum_{\ell=0}^{d} (-1)^\ell \beta^\ell.$$

The component of Chow degree d gives

$$\begin{aligned}\mathrm{DMV}(\Delta_{I_1}, \dots, \Delta_{I_k}) &= (-1)^d(-1)^k(-1)^{d-k} \int_{X_E} h_{I_1} \cdots h_{I_k} \beta^{d-k} \\ &= \int_{X_E} h_{I_1} \cdots h_{I_k} \beta^{d-k}.\end{aligned}$$

The classes h_{I_j} and β are the nef toric divisor classes associated to Δ_{I_j} and ∇_E, respectively. This is exactly the standard intersection-theoretic expression for the mixed volumes, i.e.,

$$\int_{X_E} h_{I_1} \cdots h_{I_k} \beta^{d-k} = d!\, \mathrm{MV}\big(\Delta_{I_1}, \dots, \Delta_{I_k}, \nabla_E[d-k]\big),$$

which proves (6). □

Characteristic coefficients of cotransversal matroids. To give a matroidal interpretation of this inequality, we use the results of [DRHN19]. Let Λ be a lattice of rank d, let $P_1, \dots, P_k \subseteq \Lambda_{\mathbb{R}}$ be lattice polytopes, and let

$$Y \subseteq T_\Lambda := \operatorname{Spec} \mathbb{C}[\Lambda]$$

be the complete intersection defined by generic Laurent polynomials $f_1, \dots, f_k$ with Newton polytopes $P_1, \dots, P_k$.

For a complex quasi-projective variety Y, denote by $h^{p,q}\big(H_c^\ell(Y)\big)$ the Hodge numbers of the mixed Hodge structure on $H_c^\ell(Y, \mathbb{Q}) \otimes \mathbb{C}$, and set

$$e^{p,q}(Y) := \sum_{\ell \geq 0} (-1)^\ell h^{p,q}\big(H_c^\ell(Y)\big).$$

The Hodge–Deligne polynomial of Y is

$$E(Y;u,v) := \sum_{p,q} e^{p,q}(Y) u^p v^q,$$

and the motivic arithmetic genus of Y is

$$(-1)^{\dim Y} E(Y;0,1).$$

By [DRHN19, Theorem 1.6],

$$\mathrm{DMV}(P_1,\dots,P_k) = (-1)^{\dim Y} E(Y;0,1). \tag{7}$$

If $Y = \varnothing$, then both the Hodge–Deligne polynomial and the corresponding discrete mixed volume vanish, so from now on we assume that Y is nonempty.

We now apply this identity to the coordinate simplices introduced above.

Proposition 4.5. *Let M be a loopless cotransversal matroid on*

$$E = \{0,1,\dots,d\}$$

of rank $1 \le r \le d$. Choose a transversal presentation $I_1,\dots,I_k \subseteq E$ of M^* with $k = d+1-r$, and set $\Delta_j := \Delta_{I_j}$. Write

$$\overline{\chi}_M(q) := \frac{\chi_M(q)}{q-1} = \sum_{i=0}^{r-1} (-1)^i \mu_i(M) q^{r-1-i}$$

for the reduced characteristic polynomial of M. Then, for $0 \le i \le r-1$,

$$\mu_i(M) = d!\,\mathrm{MV}\big(\Delta_1,\dots,\Delta_k,\Delta_E[r-1-i],\nabla_E[i]\big). \tag{8}$$

Consequently, the reduced and unreduced characteristic coefficients of M are log-concave, as proved for this case first in [Huh12] *and for all matroids in* [AHK18].

Proof. Let $A = (a_{ji})$ be a generic $k \times (d+1)$ matrix, with columns indexed by E, whose j-th row has support I_j, and put

$$W := \ker(A) \subseteq \mathbb{C}^E.$$

A minor of A is generically nonzero precisely when the corresponding columns can be matched to distinct rows. Hence the column matroid of A is the transversal matroid presented by $I_1,\dots,I_k$, namely M^*. The coordinate restrictions $x_i|_W \in W^\vee$ for $i \in E$ therefore represent M. Recall that

$$T_E := \operatorname{Spec}\mathbb{C}[M_E] \cong (\mathbb{C}^*)^E/\mathbb{C}^* \subseteq \mathbb{P}(\mathbb{C}^E).$$

Since the only lattice points of Δ_{I_j} are its vertices, a generic Laurent polynomial with Newton polytope Δ_{I_j} has the form

$$f_j = \sum_{i \in I_j} a_{ji} \chi^{e_i - e_0}.$$

In homogeneous coordinates $[x_i]_{i\in E}$ on T_E, one has

$$\chi^{e_i-e_0} = \frac{x_i}{x_0}.$$

Since x_0 is invertible on T_E, the equation $f_j = 0$ is equivalent to the homogeneous linear equation

$$\sum_{i\in I_j} a_{ji}x_i = 0.$$

Thus the corresponding complete intersection is

$$Y = \mathbb{P}(W) \cap T_E = \mathbb{P}(W) \setminus \bigcup_{i\in E} \mathbb{P}\big(W \cap \{x_i = 0\}\big).$$

This is the projective complement of the coordinate hyperplanes restricted to W, and the defining linear forms $x_i|_W$, $i \in E$, represent M. If some of the resulting hyperplanes coincide, removing the repeated hyperplanes gives the simplification of M. This does not affect the characteristic polynomial, since simplification preserves the lattice of flats.

By [Alu13, Theorem 1.1], the class of this projective arrangement complement in the Grothendieck ring of varieties is

$$[Y] = \overline{\chi}_M([\mathbb{A}^1]).$$

The Hodge–Deligne polynomial is additive for closed–open decompositions and multiplicative for products, and thus induces a ring homomorphism $E(-;u,v) : K_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}[u,v]$ which sends the class $\mathbb{L} = [\mathbb{A}^1]$ to uv. Applying this homomorphism to Aluffi's identity $[Y] = \overline{\chi}_M(\mathbb{L})$ therefore gives

$$E(Y;u,v) = \overline{\chi}_M(uv).$$

Since $\dim Y = r-1$, equation (7) yields

$$\mathrm{DMV}(\Delta_1,\dots,\Delta_k) = (-1)^{r-1}\overline{\chi}_M(0) = \mu_{r-1}(M). \tag{9}$$

For $0 \le m \le r-1$, define

$$a_m := \mathrm{DMV}(\Delta_1,\dots,\Delta_k,\Delta_E[m]).$$

Adding one copy of Δ_E amounts to adding one generic full-support homogeneous linear equation. Hence adding m copies of Δ_E replaces W by a generic codimension-m subspace $W_m \subseteq W$. Let M_m be the matroid represented by the coordinate restrictions on W_m. We claim that $M_m = \mathrm{Tr}^m(M)$, the m-fold truncation of M. Let

$$K_m := \ker\big(W^\vee \longrightarrow W_m^\vee\big).$$

Then K_m is a generic m-dimensional subspace of the r-dimensional space $W^\vee$. For $S \subseteq E$, set $U_S := \mathrm{span}\{x_i|_W : i \in S\} \subseteq W^\vee$. Genericity gives $\dim(U_S \cap K_m) =$

$\max\{0, \dim U_S + m - r\}$, and therefore

$$\begin{aligned}\mathrm{rk}_{M_m}(S) &= \dim \mathrm{im}(U_S \to W_m^\vee) \\ &= \dim U_S - \dim(U_S \cap K_m) \\ &= \min\{\mathrm{rk}_M(S), r - m\},\end{aligned}$$

which is precisely the rank function of $\mathrm{Tr}^m(M)$.

Now, applying (9) to $\Delta_1, \dots, \Delta_k, \Delta_E[m]$ gives

$$a_m = \mu_{r-m-1}(M_m).$$

Truncation preserves the flats of rank less than $r - m$, together with the corresponding intervals in the lattice of flats. To make the resulting coefficient shift explicit, write

$$\chi_N(q) = \sum_{\ell=0}^{\mathrm{rk}\, N} (-1)^\ell w_\ell(N) q^{\mathrm{rk}\, N - \ell}$$

for any loopless matroid N. Then $w_\ell(M_m) = w_\ell(M)$ for $\ell < r - m$. Since $\mu_j(N) = \sum_{\ell=0}^{j} (-1)^{j-\ell} w_\ell(N)$, it follows that $\mu_j(M_m) = \mu_j(M)$ for $0 \le j \le r - m - 1$. Consequently,

$$a_m = \mu_{r-m-1}(M). \tag{10}$$

On the other hand, Theorem 4.4 gives

$$\begin{aligned}a_m &= d!\, \mathrm{MV}\big(\Delta_1, \dots, \Delta_k, \Delta_E[m], \nabla_E[d - (k+m)]\big) \\ &= d!\, \mathrm{MV}\big(\Delta_1, \dots, \Delta_k, \Delta_E[m], \nabla_E[r - 1 - m]\big),\end{aligned}$$

as $k = d + 1 - r$. Substituting $m = r - 1 - i$ and using (10) proves (8). Note that the number of mixed-volume arguments is $k + (r - 1 - i) + i = d$, as expected. Log-concavity then follows from the Alexandrov–Fenchel inequality, applied to the moving bodies Δ_E and ∇_E. □

We note that the coefficients of the unreduced characteristic polynomial $\chi_M(q) = \sum_{j=0}^{r} (-1)^j w_j(M) q^{r-j}$ are also mixed volumes, which follows from $w_j(M) = \mu_j(M) + \mu_{j-1}(M)$, the mixed volume expression for $\mu_j(M)$, and the multilinearity of mixed volumes, which gives

$$w_j(M) = d!\mathrm{MV}\big(\Delta_1, \dots, \Delta_k, \Delta_E[r - 1 - j], \nabla_E[j - 1], \Delta_E + \nabla_E\big)$$

for $1 \le j \le r - 1$. The log-concavity of $(w_j(M))$ also follows directly from this sequence being the convolution of $(\mu_i(M))$ with $(1, 1)$, and the fact that nonnegative log-concave sequences without internal zeros are closed under convolution.

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