Master's Programme in Mathematics and Operations Research

# Nonpositive curvature, $\ell^2$-invariants and right-angled Coxeter groups

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Aalto University

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# Nonpositive curvature, $\ell^2$-invariants and right-angled Coxeter groups

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## Abstract

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### Abstract

The Hopf conjecture predicts that the Euler characteristic of a closed $2k$-manifold $M$ with nonpositive sectional curvature satisfies $(-1)^k\chi(M) \geq 0$. Following the introduction of $\ell^2$-methods into topology, Atiyah proved that the Euler characteristic of a closed manifold can be computed using the $\ell^2$-cohomology of its universal cover. With this formula, Dodziuk and Singer realized that there is a stronger conjecture about the $\ell^2$-cohomology of the universal covers of aspherical manifolds, which predicts that the $\ell^2$-cohomology of such spaces vanishes outside the middle dimension. We present Gromov's proof of this conjecture for Kähler hyperbolic manifolds and Dodziuk's proof for rotationally symmetric spaces, including hyperbolic space.

The Hopf conjecture can also be studied for cell complexes with nonpositive curvature in the CAT(0)-sense. By a result of Gromov, a space cellulated by regular Euclidean cubes is a manifold of nonpositive curvature if and only if the links of vertices are flag-triangulated spheres. A combinatorial analogue of the Gauss-Bonnet theorem allows us to compute the Euler characteristic as a sum depending only on the face numbers of these flag spheres, and in fact the conjecture is equivalent to these local contributions having the correct sign. This statement is the well-known Charney-Davis conjecture in combinatorics. The equivalence is due to a construction known as the Danzer complex $\mathfrak{D}(L)$, which is a cubical complex with all vertex links isomorphic to a fixed simplicial complex $L$. We show how the homology of the Danzer complex can be decomposed into the homology of induced subcomplexes of $L$, and using this, formulate the Charney-Davis conjecture in terms of the Betti numbers of the Stanley-Reisner ring of $L$.

As in the smooth case, one may investigate the Dodziuk-Singer conjecture for these aspherical cubical complexes. When $L$ is a flag complex, the universal cover of $\mathfrak{D}(L)$ can be explicitly constructed using a right-angled Coxeter group associated to $L$. This space is known as the Davis complex $\Sigma_L$ of $L$. We present a proof of the vanishing of the $\ell^2$-cohomology of $\Sigma_L$ outside the middle degree in dimensions up to four, due to Davis and Okun. Their arguments rely on inductive methods and powerful results on hyperbolic 3-manifolds, culminating in a proof of the Charney-Davis conjecture for flag 3-spheres.

**Keywords** Hopf conjecture, nonpositive curvature, $\ell^2$-cohomology, Right-angled Coxeter groups

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**Tekijä**
Joel Hakavuori

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**Tiivistelmä**

Hopfin konjektuuri ennustaa, että asfäärisen suljetun $2k$-moniston $M$ Eulerin karakteristiikka toteuttaa epäyhtälön $(-1)^k\chi(M) \geq 0$. Cartan-Hadamard -teoreema takaa, että monistot, joilla on ei-positiivinen sektionaalinen kaarevuus ovat asfäärisiä, minkä johdosta konjektuuria tutkitaan usein geometrisesta näkökulmasta. Atiyah on osoittanut, että suljetun moniston Eulerin karakteristiikka voidaan laskea käyttämällä sen universaalin peitteen $\ell^2$-kohomologiaa. Tämän tuloksen perusteella Dodziuk ja Singer muodostivat Hopfin konjektuuria vahvemman väitteen, jonka mukaan asfäärisen suljetun moniston universaalin peitteen $\ell^2$-kohomologia on triviaali keskimmäisen asteen ulkopuolella. Esitämme Gromovin todistuksen tästä konjektuurista hypebolisille Kähler-monistoille sekä Dodziukin todistuksen rotaatiosymmetrisille monistoille, mukaan lukien hyperboliselle avaruudelle.

Hopfin konjektuuria voidaan tutkia myös solukomplekseille joilla on ei-positiivinen kaarevuus CAT(0)-mielessä, jotka voidaan osoittaa asfäärisiksi kuten sileässä tapauksessa. Gromov on todistanut, että Euklidisten kuutioiden soluttamalla avaruudella on ei-positiivisen kaarevuus jos ja vain jos solmujen linkit ovat lippukolmiotettuja palloja. Gauss-Bonnet-lauseen kombinatorinen versio mahdollistaa Eulerin karakteristiikan laskemisen summana, joka riippuu vain näiden lippupallojen kombinatoriikasta. Hopfin konjektuuri Euklidisten kuutioiden soluttamille monistoille on tässä kontekstissa ekvivalentti väitteelle, että näiden paikallisten summien etumerkki on oikea. Tämä väite tunnetaan Charney-Davis -konjektuurina kombinatoriikassa. Ekvivalenssi seuraa Danzerin kompleksista $\mathfrak{D}(L)$, joka on kuutioitu solukompleksi, jonka kaikki solmulinkit ovat isomorfisia simplisen kompleksin $L$ kanssa. Näytämme, kuinka Danzerin kompleksin homologia voidaan hajottaa solmujen linkkien indusoitujen alikompleksien homologiaksi. Käyttämällä tätä tulosta, muotoilemme Charney-Davis -konjektuurin väitteenä $L$:n Stanley-Reisner -renkaan Betti-luvuista.

Kuten sileässä tapauksessa, voimme tutkia Dodziuk-Singer -konjektuurin paikkaansapitävyyttä näiden asfääristen solukompleksien universaaleille peitteille. Kun $L$ on lippukompleksi, sen Danzerin kompleksin universaalista peitteestä on olemassa eksplisiittinen rakennelma, joka on $L$:n Davisin kompleksi $\Sigma_L$. Tämä tila rakennetaan käyttämällä $L$:stä muodostettua suorakulmaista Coxeter-ryhmää. Esitämme Davisin ja Okunin todistuksen $L$:n Davisin kompleksin $\ell^2$-kohomologian katoamisesta keskimmäisen asteen ulkopuolella neljään ulottuvuuteen asti. Heidän väitteensä perustuvat induktiivisiin menetelmiin ja vahvoihin tuloksiin hyperbolisista kolmiulotteisista monistoista, mikä huipentuu Charney-Davis -konjektuurin todistukseen lippukolmioiduille kolmiulotteisille palloille.

**Avainsanat** Hopfin konjektuuri, ei-positiivinen kaarevuus, $\ell^2$-kohomologia, oikeakulmaiset Coxeter-ryhmät

## Preface

I would like to express my deepest gratitude to my advisor, Prof. Karim Adiprasito, for generously sharing his expertise and for his guidance throughout the thesis process, beginning with suggesting the topic. I also thank him for organizing a visit to the Institut de Mathématiques de Jussieu-Paris Rive Gauche, which greatly assisted in the writing of this thesis.

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# Contents

# Introduction

In the spring of 1931, Heinz Hopf gave a talk in Fribourg, Switzerland, where he conjectured that the Euler characteristic of a closed Riemannian $2k$-manifold $M$ with negative sectional curvature satisfies the inequality

$$(-1)^k \chi(M) > 0.$$

In the modern, purely topological formulation of the conjecture, it is stated that if $M$ is an aspherical closed $2k$-manifold, then $(-1)^k \chi(M) \geq 0$. This generalizes Hopf's original conjecture, as the Cartan-Hadamard theorem guarantees that the universal cover of any Riemannian manifold with nonpositive sectional curvature is contractible. In dimensions 2 and 4, the conjecture can be proved using the Chern-Gauss-Bonnet theorem, as in these cases the integrand has the correct sign [8]. However, as shown by Geroch in [16], this method fails in higher dimensions.

In 1976, Atiyah [4] introduced the use of $\ell^2$-methods in topology. For a finite $CW$-complex $X$, one uses infinite chains with square-summable coefficients on the universal cover $\widetilde{X}$ of $X$ to define the $\ell^2$-cohomology groups $H^i_{(2)}(\widetilde{X})$, and if $M$ is a compact Riemannian manifold, one applies de Rham cohomology to square-integrable differential forms on the universal cover $\widetilde{M}$ to define $H^i_{(2)}(\widetilde{M})$. These cohomology groups, if nonzero, are typically infinite-dimensional Hilbert spaces. However, using the fact that the fundamental group $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformations, one can normalize the dimension to associate a nonnegative real number to the spaces $H^i_{(2)}(\widetilde{M})$, known as the $\ell^2$-Betti numbers $b_i^{(2)}(\widetilde{M})$. By a result of Atiyah, the Euler characteristic of $M$ can be computed using the $\ell^2$-Betti numbers as $\chi(M) = \sum_{i=0}^{2k} (-1)^i b_i^{(2)}(\widetilde{M})$. Dodziuk, based on a suggestion of Singer, investigated in [13] how the Hopf conjecture could be approached using Atiyah's formula for $\chi(M)$. This led to the Dodziuk-Singer conjecture, which predicts that for a closed aspherical $n$-manifold $M$,

$$H^i_{(2)}(\widetilde{M}) = 0 \quad \text{for all} \quad i \neq \frac{n}{2},$$

which is equivalent to the vanishing of the $\ell^2$-Betti numbers outside the middle degree. In Chapter 1, we study the Dodziuk-Singer conjecture in the context of smooth manifolds, presenting a proof of the conjecture for Kähler hyperbolic manifolds following the work of Gromov [18], and for rotationally symmetric manifolds based on the work of Dodziuk [13].

The focus of Chapter 2 is studying the Hopf and Dodziuk-Singer conjectures in the context of CAT(0)-spaces, which generalize the notion of nonpositive curvature to metric spaces. This approach was initiated by Charney and Davis in [7], where they studied the Hopf conjecture for piecewise Euclidean manifolds. When the manifold possesses a cellular structure constructed by gluing regular Euclidean cubes along isometries of their faces, by a lemma of Gromov the condition of nonpositive curvature is equivalent to a purely combinatorial statement, namely that the link of each vertex is a flag simplicial complex. Such spaces are aspherical by a metric version of the Cartan-Hadamard theorem, and hence cube complexes whose vertex links are flag spheres give examples of closed, aspherical manifolds.

There is a combinatorial analogue of the Gauss-Bonnet theorem, which for a cubical complex $X$ simplifies to $\chi(X) = \sum_{v \text{ vertex of } X} \kappa(\mathrm{lk}_X(v))$, where $\mathrm{lk}_X(v)$ is the link of the vertex $v$ in $X$ and $\kappa(\Delta) = 1 + \sum i = 0^{\dim \Delta} \left(-\frac{1}{2}\right)^{i+1} f_i(\Delta)$, where $f_i(\Delta)$ are the face numbers of $\Delta$. In contrast to the smooth case, the Hopf conjecture for cube complexes is equivalent to the local contributions $\kappa(\mathrm{lk}_X(v))$ having the correct sign. This is due to a construction of Danzer [10], which shows that one can form a cubical complex $\mathfrak{D}(L)$ whose all vertex links are isomorphic to a fixed simplicial complex $L$. Hence, in this context, the Hopf conjecture becomes a purely combinatorial problem about face enumeration of flag spheres, which is known as the Charney-Davis conjecture in combinatorics. As the Euler characteristic of $\mathfrak{D}(L)$ and $\kappa(L)$ have the same sign, proving that $\mathfrak{D}(L)$ satisfies the Hopf conjecture when $L$ is a flag complex is equivalent to the Charney-Davis conjecture. Motivated by this, we prove decomposition theorems for the homology groups of $\mathfrak{D}(L)$, which breaks $H_*(\mathfrak{D}(L))$ down into components coming from induced subcomplexes of $L$. Using these results, we formulate the Charney-Davis conjecture in terms of the graded Betti numbers of the Stanley-Reisner ring of $L$.

As in the case of smooth manifolds, one may consider the Dodziuk-Singer conjecture on the $\ell^2$-cohomology of the universal cover of $\mathfrak{D}(L)$. When $L$ is a flag complex, an explicit construction of the universal cover of $\mathfrak{D}(L)$ is given by the Davis complex $\Sigma_L$, which is constructed as the order complex of a poset of subgroups of a right-angled Coxeter group associated to $L$. We present a proof of the vanishing of the $\ell^2$-cohomology of $\Sigma_L$ outside the middle degree in dimensions up to four, following Davis and Okun [12]. Their arguments rely on inductive methods and powerful results on hyperbolic 3-manifolds, culminating in a proof of the Charney-Davis conjecture for flag 3-spheres.

# Chapter 1

# The Hopf and Dodziuk-Singer conjectures

Chapter 1 of this thesis is dedicated to studying the Hopf and Dodziuk-Singer conjectures for smooth manifolds. In Section 1, we cover the necessary background in geometry and topology, define the $\ell^2$-cohomology groups of a smooth manifold and present the Hopf and Dodziuk-Singer conjectures. In Section 2, we present Gromov's proof of the conjectures for Kähler hyperbolic manifolds, and in Section 3 we cover Dodziuk's proof of the conjectures for rotationally symmetric manifolds.

## 1 Preliminaries and the main conjectures

The aim of this section is to present some of the background and basic results in topology and geometry that we will need in the coming sections, as well as give the basic analytic definition of $\ell^2$-cohomology that we will use in Sections 2 and 3. We expect the reader to be familiar with basic algebraic topology and differential geometry.

### 1.1 Topology

Let us start by recalling some basic definitions of different models of cell complexes and their topology. For more details and proofs of the presented claims, we refer to [19], or any other basic text in algebraic topology.

Let $B^n$ be the closed unit ball in $\mathbb{R}^n$, so that $\partial B^n = \mathbb{S}^{n-1}$. An *$n$-cell* $\sigma$ is a topological space homeomorphic to $B^n$, and a cell is a space which is homeomorphic to $B^n$ for some $n \in \mathbb{N}$. We denote by $\dot{\sigma}$ the subspace of $\sigma$ which corresponds to $\mathbb{S}^{n-1} \subset B^n$ under any homeomorphism $\sigma \to B^n$. Given a cell $\sigma$ and a continuous map $f : \dot{\sigma} \to X$ for some topological space $X$, we may form the space $X \cup_f \sigma = (X \bigsqcup \sigma)/\sim$, where $\sim$ is the equivalence relation $p \sim f(p)$ for all $p \in \dot{\sigma}$. The map $f$ is referred to as an *attaching map*.

A *CW complex* or *cell complex* is a topological space $X$ built by gluing $n$-cells to the $(n-1)$-skeleton $X^{(n-1)}$ of $X$; we start with a discrete set $X^{(0)}$ of points, regarded as 0-cells. Then, inductively form the $n$-skeleton $X^{(n)}$ of $X$ by attaching $n$-cells to $X^{(n-1)}$ via attaching maps. If $X = X^{(n)}$ for some $n \in \mathbb{N}$, then $X$ is an $n$-dimensional cell complex. A CW complex is *regular* if the attaching maps are embeddings. For an arbitrary topological space $X$, a *cell decomposition* is a family $\{\sigma_\alpha \mid \alpha \in I\}$ of cells $\sigma_\alpha \subseteq X$ so that $X = \bigsqcup_{\alpha \in I} \sigma_\alpha$. The *$k$-skeleton* $X^{(k)}$ of $X$ is the subspace consisting of cells of dimension $\leq k$. A continuous map $f : X \to Y$ of CW-complexes is *cellular* if $f(X^{(k)}) \subset Y^{(k)}$ for all $k$. A *subcomplex* of a cell complex $X$ is a closed subspace $A \subset X$, such that $A$ is a cell complex that is the union of cells of $X$. A *CW pair* $(X, A)$ is a subcomplex $A \hookrightarrow X$ and a cell complex $X$.

Simplicial complexes are a purely combinatorial model of cell complexes, and we will use them extensively throughout the thesis. An *abstract simplicial complex* $(V, \Delta)$, or simply $\Delta$, for a set $V$, is a collection $\Delta$ of finite non-empty subsets of $V$, such that for every $A$ in $\Delta$ and non-empty subset $B \subseteq A$, $B$ is also in $\Delta$. The elements of $V = V(\Delta)$ are the *vertices* of $\Delta$ and the finite sets in $\Delta$ are called the *faces* or *simplices* of $\Delta$. When $V$ is finite we choose an identification with $[n] = \{1, \ldots, n\}$, where $n = |V|$. A complex $(V, \Delta)$ is finite if the set of vertices $V$ is finite. The dimension of a face $\sigma \in \Delta$ is $|\sigma| - 1$, and the dimension of a complex is the maximal dimension of a face in the complex. A *simplicial map* $f : K \to L$ is a map from $V(K)$ to $V(L)$ that maps every simplex in $K$ to a simplex in $L$.

A $k$-simplex $\Delta^k$ is a $k$-dimensional polytope which is the convex hull of $k+1$ affinely independent points $v_0, \ldots, v_k$, the vertices of $\Delta^k$. Then standard $k$-simplex $\Delta^k$ is given by $\Delta^k = \{(t_0, \ldots, t_k) \mid t_i \geq 0, \ \sum_{i=0}^{k} t_i = 1\} \subset \mathbb{R}^{k+1}$. A *geometric simplicial complex* $\Delta$ is a collection of simplices such that every face of a simplex is also in $\Delta$, and the non-empty intersection of any two simplices $P, L \in \Delta$ is a face of both $P$ and $L$. The dimension of $\Delta$ is the same as in the abstract case, and the $k$-skeleton $\Delta^{(k)}$ of $\Delta$ is the subspace of $\Delta$ consisting of all simplices of dimension $k$ and less. Clearly, every geometric simplicial complex defines an abstract one, and every abstract simplicial complex has a geometric realization. For an abstract simplicial complex $K$, we denote its geometric realization by $|K|$. From now on, we simply refer to simplicial complexes. A *subdivision* of a simplicial complex $K$ is a simplicial complex $L$ such that $|K| = |L|$ and each simplex of $L$ is contained in some simplex of $K$. A *triangulation* of a topological space $X$ is a (geometric) simplicial complex $|K|$ together with a homeomorphism $|K| \cong X$. A topological space $X$ is *triangulable* if it has a triangulation.

For a simplicial complex $K$ and a commutative ring $R$, we let $C_i(K, R)$ denote the $i^{th}$ *chain group* of $K$, which is the free $R$-module generated by the $i$-simplices in $K$. For a chain $\tau = \sum \lambda_i \sigma_i$, $\lambda_i \in R$, we let $|\tau|$ denote the *support of* $\tau$, which is the subspace of $K$ consisting $\sigma_i$ with $\lambda_i \neq 0$. The standard choice of coefficients is $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ or a field of characteristic zero. With coefficients in a ring such as $\mathbb{Z}$, the homology groups may contain a torsion part, but this vanishes when working working over a field of characteristic zero. From now on, unless otherwise mentioned, we work with coefficients in a field $k$ of characteristic zero, such as $\mathbb{R}$, and drop the notation specifying the choice of coefficients.

The *simplicial boundary* $\partial : C_i(K) \to C_{i-1}(K)$ is the linear extension of the map defined by $\partial([v_0, \ldots, v_i]) = \sum_{j=0}^{i}(-1)^j [v_0, \ldots, \hat{v_j}, \ldots, v_k]$ on the basis of $C_i(K)$ for some fixed ordering of the vertices. Define $Z_i(K) := \ker\ (\partial : C_i(K) \to C_{i-1}(K))$ to be the subspace of $i$-cycles and $B_i(K) := \mathrm{im}\ (\partial : C_{i+1}(K) \to C_i(K))$ to be the $i$-boundaries. The simplicial boundary satisfies $\partial^2 = 0$, so we may define the $i^{th}$ *simplicial homology group* $H_i(K)$ to be the $i^{th}$ homology group of the chain complex $(C_\bullet(K), \partial)$, i.e., the quotient $Z_i(K)/B_i(K)$. We write $H_i(K) = \langle \zeta_1, \ldots, \zeta_m \rangle$ if the $i^{th}$ homology group is generated by some $i$-cycles $\zeta_1, \ldots \zeta_m \subset C_i(K)$. We set $H_*(K) = \oplus_i H_i(K)$.

Simplicial homology is a topological invariant of the space $|K|$; if $|L|$ is a simplicial complex homotopy equivalent to $|K|$, then they have isomorphic homology groups. The $i^{th}$ *Betti number* $b_i(K)$ of $K$ is the $k$-dimension of $H_i(K)$, or equivalently the rank of the free part of $H_i(K, \mathbb{Z})$. The *reduced homology* $\tilde{H}_*(K)$ of $K$ is defined by adding the augmentation map $\epsilon : C_0(K) \to k,\ \epsilon : (\sum \lambda_i v_i) = \sum \lambda_i$ to the simplicial chain complex of $K$, and we have $H_i(K) \cong \tilde{H}_i(K)$ for all $k \geq 1$, and $\bar{b}_0(K) = \dim_k \tilde{H}_i(K) = b_0(K) - 1$.

For triangulable topological spaces, such as smooth manifolds, their homology is defined as the simplicial homology of some triangulation. For non-triangulable spaces we use singular or cellular homology [19, Ch. 2.2], and all three of these options give isomorphic homology groups. To define singular homology, let $X$ be any topological space. A *singular $i$-simplex* $\sigma$ in $X$ is a continuous map $\sigma : \Delta^i \to X$ of the standard $k$-simplex into $X$. If $\sigma = [v_0, \ldots, v_i]$ denote the vertices of $\sigma$, the boundary $\partial\sigma$ of $\sigma$ is given by $\partial\sigma = \sum_{j=0}^{i}(-1)^j [v_0, \ldots, \hat{v_j}, \ldots, v_i]$, which again satisfies $\partial^2 = 0$. We define $C_i(X)$ to be the free abelian group over the set of all singular $i$-simplices in $X$, and extend the boundary $\partial$ linearly to $C_i(X)$. The homology of the singular chain complex $(C_\bullet(X), \partial)$ are the *singular* *homology* groups of $X$. As in the simplicial case, we may change coefficients by defining $C_i(X, R)$ to be the free $R$-module over $i$-simplices in $X$ to get the singular homology groups $H_k(X, R)$ with coefficients in $R$ (with $R = \mathbb{Z}$ giving the above definition). For a pair of space $(X, A)$, in particular for CW pairs, we can compute the *relative homology* $H_*(X, A)$ of $(X, A)$ by quotienting out $C_i(A)$ from $C_i(X)$ and computing homology.

Dualizing either of the above approaches leads to simplicial or singular *cohomology*. We denote the $i^{th}$ *cochain group* by $C^i(X) = \hom_k(C_i(X), k)$, the space of $k$-linear maps from $C_i(X)$ to $k$. The *coboundary map* $\delta : C^i(X) \to C^{i+1}(X)$ is given by $\delta : \varphi \mapsto \varphi \circ \partial$, which is dual to $\partial$, again satisfying $\delta^2 = 0$. Defined analogously to the case of homology, $H^i(X)$ is the $i^{th}$ *cohomology group*. Unlike in the case of homology, the cohomology groups have an additional structure of a graded ring $H^*(X) = \oplus_i H^i(X)$, where the product operation is given by the *cup product* $\smile$: $H^p(X) \times H^q(X) \to H^{p+q}(X)$. If $\sigma : \Delta^{p+q}$ is a singular $(p+q)$-simplex in $X$ and $\varphi \in \hom(C_p(X), k), \psi \in \hom(C_q(X), k)$, then $(\varphi \smile \psi)(\sigma) = \varphi(\sigma |_{[v_0,\dots,v_p]})\psi(\sigma |_{[v_p,\dots,v_{p+q}]})$, where $\sigma |_{[v_i,\dots,v_{i+k}]}$ denotes the restriction to the face of $\sigma$ on the specified vertices. The cup product satisfies the coboundary formula $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^p \varphi \smile \delta\psi$, so that we get an induced map on the cohomology groups.

For both homology and cohomology, we have the standard homological tools such as the Mayer-Vietoris sequence, Künneth theorem, excision and the long exact sequences of pairs and triples. Analogous results hold for cellular $\ell^2$-cohomology, which we will present in Section 6.

Fixing some base point $x_0 \in X$, the *fundamental group* $\pi_1(X)$ of $X$ consists of the homotopy classes of loops $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, with the group operation being concatenation of loops. The higher homotopy groups $\pi_n(X, x_0)$ are defined as homotopy classes of maps $f : \mathbb{S}^n \to X$, with $f : (s_0) = x_0$ for some fixed point $s_0 \in \mathbb{S}^n$. All homotopy groups are independent of the choice of base point, i.e., if $x, x' \in X$, then there exists an isomorphism $\phi : \pi_n(X, x) \xrightarrow{\sim} \pi_n(X, x')$, so we may speak of the homotopy groups of $X$.

For a continuous map of topological spaces $f : X \to Y$, we have induced maps $f_* : H_*(X) \to H_*(Y)$, $f^* : H^*(Y) \to H^*(X)$ and $f_* : \pi_*(X) \to \pi_*(Y)$.

**Definition 1.1.1.** The *Euler characteristic* $\chi(X)$ of a topological space $X$ is given by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i b_i(X),$$

where $b_i(X) = \text{rank } H_i(X, \mathbb{Z})$ is the $i^{th}$ Betti number of $X$.

The Euler characteristic may equivalently be defined as the alternating sum $\sum_{i=0}^{n}(-1)^i f_i(K)$ of the face numbers of some triangulation $|K|$ of $X$, or the number of $i$-cells in a finite CW-complex homotopy equivalent to $X$.

### 1.2 Geometry

In this section we recall without proof some basic notions and results in differential geometry and the topology of manifolds. Everything here can be found in, e.g., [21] or [28]. A *manifold* is a second countable Hausdorff space that is locally homeomorphic to Euclidean space, and a *manifold with boundary* is a second countable Hausdorff space that is locally homeomorphic to Euclidean space or to a half-space $\{x \in \mathbb{R}^n \mid x_n \geq 0\}$. A *smooth manifold* is a manifold with a $C^\infty$ atlas, and a *Riemannian manifold* $(M, g)$ is a smooth manifold $M$ equipped with a positive-definite inner product $g_p$ on every tangent space $T_pM$, varying smoothly in $p$. We denote the space of smooth $\mathbb{R}$-valued functions on $M$ by $C^\infty(M, \mathbb{R})$, $k$-forms on $M$ by $\Omega^k(M)$, and the Riemannian volume form in $\Omega^n(M)$ by vol.

The space of vector fields on $M$ is denoted by $\mathfrak{X}(M)$ and the tangent bundle by $TM$. An *affine connection* on $M$ is a $\mathbb{R}$-bilinear map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \mapsto \nabla_X Y$, so that for all $X, Y \in \mathfrak{X}(M)$ and all $f \in C^\infty(M, \mathbb{R})$

(i) $\nabla$ is $C^\infty(M, \mathbb{R})$-linear in $X$: $\nabla_{fX} Y = f \nabla_X Y$, and

(ii) satisfies the Leibniz rule in $Y$: $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, where $Xf$ is the directional derivative of $f$ in the direction of $X$.

The fundamental lemma of Riemannian geometry tells us that for any Riemannian manifold $(M, g)$, there exists a unique affine connection on the tangent bundle $TM$, the *Levi-Civita connection*, that satisfies the following two additional conditions:

(iv) $[X, Y] = \nabla_X Y - \nabla_Y X$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields, and

(iiv) $\nabla$ is compatible with $g$: $\nabla_X \langle X, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle X, \nabla_X Z \rangle$.

Given an affine connection $\nabla$ on $M$ and a curve $\gamma : I \to M$, there exists a unique $\mathbb{R}$-linear map $\frac{D}{dt} : \Gamma(TM\mid_{\gamma(t)}) \to \Gamma(TM\mid_{\gamma(t)})$, where $\Gamma(TM\mid_{\gamma(t)})$ denotes the space of vector fields on the curve $\gamma$, such that for all $X \in \Gamma(TM\mid_{\gamma(t)})$

- $\frac{D(fX)}{dt} = \frac{df}{dt} X + f \frac{DX}{dt}$ for all $f \in C^\infty(M, \mathbb{R})$, and

- If $X$ is the restriction of $\widetilde{X} \in \mathfrak{X}(M)$ to $\gamma(t)$, then $\frac{DX}{dt}(t) = \nabla_{\gamma'(t)}\widetilde{X}$.

The map $\frac{D}{dt}$ is the *covariant derivative* along $\gamma$ associated to the connection $\nabla$. For a curve $\gamma : I \to M$ we say that a vector field $X$ is parallel to $\gamma$ if $\frac{DX}{dt} \equiv 0$ along $\gamma$. A curve $\gamma : I \to M$ is a *geodesic* if the covariant derivative $\frac{D\gamma'(t)}{dt}$ vanishes along $\gamma$. On a Riemannian manifold with the covariant derivative induced by the Levi-Civita connection, the speed of a geodesic $||\gamma'(t)||$ is constant.

The metric $g$ of a Riemannian manifold $M$ induces a metric space structure on $M$. For a piecewise smooth curve $\gamma : [a, b] \to M$, the *length* of $\gamma$ is defined as $\ell(\gamma) = \int_a^b |\gamma'(t)|_g dt$. The *distance* $d_g(p, q)$ between two points $p, q \in M$ is defined as the infimum of $\ell(\gamma)$ over all piecewise smooth curves $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. The space $(M, d_g)$ is a metric space, and $d_g$ is the *Riemannian distance function* on $M$. For Riemannian manifolds, the topology induced by the metric $d_g$ agrees with the manifold topology of $M$. A piecewise smooth curve $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$ is *minimizing* if $\ell(\gamma) \leq \ell(\tilde{\gamma})$ for any other such curve $\tilde{\gamma}$ with the same endpoints. This is equivalent to $\ell(\gamma) = d(p, q)$. Every minimizing curve $\gamma : [a, b] \to M$ parametrized by unit speed ($||\gamma'(t)|| = 1$) is a geodesic, and conversely every geodesic is locally minimizing, i.e., every $t_0 \in (a, b)$ has a neighbourhood $U$ such that $\gamma$ restricted to $U$ is minimizing between any pair of points.

In the following sections, we will especially be focused on complete manifolds, as the $\ell^2$-cohomology of such spaces is well-behaved. A Riemannian manifold is *geodesically complete* if any maximal geodesic extends to the whole of $\mathbb{R}$, so geodesics exist for all time. The manifold $(M, g)$ is *metrically complete* if $(M, d_g)$ is complete as a metric space. Different notions of completeness for manifolds are related by the Hopf-Rinow theorem [28, Theorem 7.1].

**Theorem 1.2.1** (Hopf-Rinow Theorem)**.** *Let $(M, g)$ be a Riemannian manifold with distance function $d_g$. Then, the following are equivalent*

*(i) $M$ is geodesically complete.*

*(ii) All closed and bounded subsets of $M$ are compact.*

*(iii) $(M, d_g)$ is a complete metric space.*

A manifold $(M, g)$ satisfying any of the above conditions is a *complete manifold.*

For a tangent vector $v \in T_pM$, there exists a unique maximal geodesic $\gamma_v : I \to M$ satisfying $\gamma_v(0) = p$ and $\gamma'(0) = v$. For any $p \in M$ there exists a neighbourhood of the origin in $T_pM$ where the *exponential map* $\exp_p : v \mapsto \gamma_v(1)$ is defined and is a diffeomorphism onto its image in $M$.

**Definition 1.2.2.** Let $M$ be a Riemannian manifold with Levi-Civita connection $\nabla$. The *Riemann curvature tensor* $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemann curvature tensor is a (1, 3)-tensor field.

The main notion of curvature we are interested in is sectional curvature. For a smooth surface $S \subset \mathbb{R}^3$, $p \in S$ and $v \in T_pS$, the normal vector $\mathbf{n}$ to $T_pM$ and $v$ span a plane in $\mathbb{R}^3$, and intersecting this plane with $S$ gives a curve $\gamma_v$ in $S$. If $\gamma_v$ is parametrized by arc length, we can define its curvature $\kappa_v$ as $||\gamma_v''(t)||$. This curvature will achieve its minimum and maximum values for some $v_1, v_2 \in T_pM$, and we define the principal curvatures $\kappa_1, \kappa_2$ as the curvatures of $\gamma_{v_i}''(t)$ for $i = 1, 2$. The *Gaussian curvature* $K : S \to \mathbb{R}$ of $S$ is given by $K = \kappa_1\kappa_2$, and by Gauss' Theorema Egregium, $K$ is an intrinsic measure of curvature of $S$, meaning that it is independent of the embedding $S \hookrightarrow \mathbb{R}^3$.

Using the Gaussian curvature of surfaces, this approach can be generalized to an $n$-manifold $M$; if $\Pi \subset T_pM$ is a 2-dimensional subspace of $T_pM$ and $U \subset T_pM$ is a neighbourhood of 0 for which the exponential map is a diffeomorphism, $S_\Pi := \exp_p(\Pi \cap U)$ is a 2-dimensional submanifold of $M$ which contains $p$. Then, the sectional curvature $K(\Pi)$ can be defined as the Gaussian curvature of this embedded surface at $p$. For completeness, we include the usual definition of sectional curvature as defined by the Riemann curvature tensor.

**Definition 1.2.3.** Let $M$ be a Riemannian $n$-manifold with Riemann curvature tensor $R$. For any $p \in M$ and linearly independent $u, v \in T_pM$, the *sectional curvature* $K(u, v)$ of $M$ at $p$ for the hyperplane $\Pi$ spanned by $u, v$ is defined as

$$K(u,v) = \frac{\langle R(u,v)v, u\rangle}{||u||^2||v||^2 - \langle u, v\rangle^2}.$$

Knowledge of all sectional curvatures on $M$ fully determines the Riemann curvature tensor $R$.

A manifold $M$ has *constant sectional curvature* if $K(\Pi)$ is constant for all $p \in M$ and $\Pi \subset T_pM$. The three model spaces for constant curvature are the complete, connected, simply connected smooth Riemannian manifolds. These are spheres, Euclidean space and hyperbolic space, with positive, zero and negative sectional curvature respectively. Any complete, connected Riemannian manifold $M$ with constant sectional curvature is isometric to $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is one of the three model spaces and $\Gamma$ is a discrete subgroup of $\mathrm{Isom}(\widetilde{M})$. A complete manifold of constant sectional curvature $-1$ is a *hyperbolic manifold.*

Next, we cover some basic covering space theory.

**Definition 1.2.4.** A *covering space* of a topological space $X$ is a space $E$ and a map $q : E \to X$ such that every point $x \in X$ has an open neighbourhood $U_x$ such that $q^{-1}(U_x)$ is a disjoint countable union of open sets $\{V_\alpha\}_{\alpha \in I}$, such that the restriction $q|_{V_\alpha} : V_\alpha \to U_x$ is a homeomorphism for all $\alpha \in I$. A simply connected covering space $p : \widetilde{X} \to X$ is a *universal cover* of $X$.

The universal cover $p : \widetilde{X} \to X$ satisfies the following universal property: if $q : E \to X$ is another simply connected covering, then there exists a unique homeomorphism $f : \widetilde{X} \to E$, such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

commutes. Hence, given any other covering map $q : E \to X$, $p : \widetilde{X} \to X$ factors through $q$, making $\widetilde{X}$ into a covering space of $E$. Hence, $p : \widetilde{X} \to X$ is in some sense the maximal covering space of $X$.

For any covering $q : E \to X$, a homeomorphism $d : E \to E$ is a *deck transformation* if the diagram

$$\begin{tikzcd} E \arrow[rr, "d"] \arrow[dr, "q"'] & & E \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

commutes. By composing deck transformations of $q : E \to X$, we get the group of deck transformation $\mathrm{Deck}(q)$ acting on $E$. A *lift* of a point $x \in X$ to a covering $q : E \to X$ is any point $x_0 \in E$ such that $q(x_0) = x$. A covering space $q : E \to X$ is *normal* if for each pair of lifts $x_1, x_2 \in E$ of any $x \in X$, there exists a deck transformation $g : E \to E$ with $g(x_1) = x_2$, so that $E/\mathrm{Deck}(q) \cong X$. A covering is normal if and only if $q_*(\pi_1(E)) \subset \pi_1(X)$ is a normal subgroup, and in particular the group of deck transformations of the universal cover $p : \widetilde{X} \to X$ is isomorphic to $\pi_1(X)$.

By definition, any simply connected space is its own universal cover. A sufficient condition for the existence of a universal cover for $X$ is that $X$ is connected and locally simply connected. All connected topological manifolds and CW complexes satisfy these conditions, and from now on we assume that unless mentioned otherwise, all the spaces we work with satisfy these conditions.

A *topological group* is a topological space $G$ that is simultaneously a group such that the group action $(g,h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ are continuous maps. A *group action* of a topological group $G$ on a space $X$ is a continuous function $G \times X \to X$, denoted $(g,x) \mapsto g.x$, such that $e.x = x$ and $(gh).x = g.(h.x)$. Here $G \times X$ is equipped with the product topology. We assume that if $G$ is a discrete group, it is equipped with the discrete topology. For a point $x \in X$, $\{g.x \mid g \in G\}$ is the *orbit* of $x$ under the action of $G$, and the quotient space $X/G$ with the quotient topology is the orbit space. The action is *free* if $g.x = x$ implies $g = e$ for any $x \in X$, and *transitive* if for any $x, y \in X$ there exists some $g$ such that $g.x = y$. If there exists a unique such $g$, then the action is called *simply transitive*. A basic result in covering space theory is that if for every $x \in X$ there exists an open neighbourhood $U_x$ such that $U_x \cap g.U_x = \varnothing$ for all $g \neq e \in G$, then the quotient map is a covering map.

To consider the induced maps $q_* : \pi_k(E) \to \pi_k(X)$ of a covering $q : E \to X$ on the homotopy groups, we choose an arbitrary base point $x_0 \in X$ and a lift $\tilde{x_0} \in E$, so that $q(\tilde{x_0}) = x_0$. We have the following basic result.

**Proposition 1.2.5** ([19, Proposition 4.4.1])**.** *A covering space projection $q : (E, \tilde{x_0}) \to (X, x_0)$ induces isomorphisms $q_* : \pi_k(E) \to \pi_k(X)$ for all $k \geq 2$.*

**Definition 1.2.6.** A topological space $X$ is *aspherical* if the universal cover $\widetilde{X}$ of $X$ is contractible.

Let $X$ be a connected topological space which is homotopy equivalent to a CW complex, for example any smooth manifold. By Proposition 1.2.5 the universal covering map $\pi : \widetilde{X} \to X$ induces an isomorphism $\pi_* : \pi_k(\widetilde{X}) \xrightarrow{\sim} \pi_k(X)$ for all $k \geq 2$. By Whitehead's Theorem, weakly contractible CW complexes are contractible, so a CW complex $X$ is aspherical if and only if $\pi_k(X) = 0$ for all $k > 0$. Such spaces are also referred to as $K(\pi, 1)$ *spaces.*

As the covering map $p : \widetilde{M} \to M$ from the universal cover is a local homeomorphism, $\widetilde{M}$ inherits a Riemannian metric $p^*\tilde{g}$ from $(M, g)$, and with this metric the projection $p : (\widetilde{M}, \tilde{g}) \to (M, g)$ becomes a local isometry. Explicitly, for each $x \in \widetilde{M}$ we have $\tilde{g}_x(X, Y) = g_{p(x)}(p_* X, p_* Y)$, where $p_* : T_x\widetilde{M} \xrightarrow{\sim} T_{p(x)}M$ is the differential of $p$. For the universal cover $p : (\widetilde{M}, \tilde{g}) \to (M, g)$ with the lifted metric, $\pi_1(M)$ acts on $\widetilde{M}$ by isometries after identifying $\mathrm{Deck}(\widetilde{M})$ with $\pi_1(M)$.

This construction can be done more generally for length spaces $p : \widetilde{X} \to X$ by defining the length of a path $\gamma : [0,1] \to \widetilde{X}$ to be the length of $\pi \circ \gamma$ in $X$ ([5, Proposition 3.25]), which will be relevant for us when working in the CAT(0)-setting in Section 4.

The classical Cartan-Hadamard theorem from Riemannian geometry is our link between nonpositive sectional curvature and topology. It tells us that the universal cover of any complete connected Riemannian manifold with nonpositive sectional curvature is homeomorphic to $\mathbb{R}^n$, and hence contractible. A proof can be found in [28, Theorem 3.3].

**Theorem 1.2.7** (Cartan-Hadamard Theorem)**.** *Suppose $M$ is a complete connected Riemannian manifold of nonpositive sectional curvature. Then for any $x \in M$, the exponential map $\exp : T_xM \to M$ is a covering projection.*

In the coming sections, we will be interested in cohomological properties of manifolds as well as more general spaces with nonpositive sectional curvature. For context, we recall some basic results in de Rham cohomology and Hodge theory on compact manifolds. For an $n$-manifold $M$, let $\Omega^k(M)$ denote the space of smooth $k$-forms, $d : \Omega^k(M) \to \Omega^{k+1}(M)$ the exterior derivative and $\wedge$ the wedge product of forms. For $\alpha \in \Omega^k(M), \beta \in \Omega^l(M)$ these satisfy

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \in \Omega^{k+l}(M)$$
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

For an $n$-manifold with boundary $M$, the integration of exterior derivatives of $(n-1)$-forms is described by Stokes' theorem, which we shall need in the coming sections.

**Theorem 1.2.8** (Stokes' theorem)**.** *Let $M$ be an oriented $n$-manifold with boundary $\partial M$, and $\omega \in \Omega_c^{n-1}(M)$ a compactly supported $(n-1)$-form on $M$. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

In particular, when $\partial M = \varnothing$, then $\int_M d\omega = 0$ for compactly supported $\omega$.

As $d \circ d = 0$, $d(\Omega^{k-1}(M)) := B^k(M) \subseteq \{\omega \in \Omega^{k+1}(M) \mid d\omega = 0\} := Z^k(M)$, and we have a chain complex $(\Omega^*(M), d)$. Elements of the image $B^k(M)$ are *exact forms*, and those of the kernel $Z^k(M)$ are *closed forms*. The $k^{th}$ de Rham cohomology group of $M$ is defined as $H^k_{dR}(M) = \frac{Z^k(M)}{B^p(M)}$, which is the $k^{th}$ cohomology group of the complex $(\Omega^*(M), d)$. The classical de Rham theorem tells us that we may compute the singular cohomology groups of a closed manifold by de Rham cohomology.

**Theorem 1.2.9** (de Rham). *Let $M$ be a closed manifold. Then the map $\int : \Omega^k(M) \to C^k(M, \mathbb{R})$ given by sending $\omega$ to $(\int_\bullet \omega : S_k(M, \mathbb{R}) \to \mathbb{R})$, the cochain mapping a singular simplex $\sigma$ to $\int_\sigma \omega$, induces an isomorphism $\int_* : H^k_{dR}(M) \xrightarrow{\sim} H^k(M, \mathbb{R})$.*

On a Riemannian $n$-manifold $M$, we have the Hodge star operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ which is uniquely determined by the condition $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle_g \mathrm{vol}$. Here $\langle \alpha, \beta \rangle_g = \langle \alpha, \beta \rangle$ is the pointwise inner product on $\Lambda^k T^*_p M$ induced by the metric, which is explicitly given by $\langle \alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta_k \rangle = \det(\langle \alpha_i^\sharp, \beta_j^\sharp \rangle_g)$, where $\alpha^\sharp$ is the vector in $T_p M$ characterized by $g_p(\alpha^\sharp, v) = \alpha(v)$.

The star operator satisfies $\star^2 \omega = (-1)^{k(n-k)} \omega$ for any $\omega \in \Omega^k(M)$ on an $n$-manifold $M$. When $M$ is compact, we have an $L^2$-inner product $(\cdot, \cdot)$ on $k$-forms given by

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle \mathrm{vol} = \int_M \alpha \wedge \star\beta.$$

This defines an $L^2$-norm on forms given by $||\alpha||^2 = (\alpha, \alpha)$. The codifferential $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is the adjoint of $d$ with respect to $(\cdot, \cdot)$, i.e., $(d\alpha, \beta) = (\alpha, \delta\beta)$ for all $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$. The codifferential on $k$-forms can be explicitly defined as $\delta = (-1)^{n(k+1)+1} \star d \star$. Then, when $M$ is closed, for $k$-forms $\alpha, \beta$ we have $\int_M d(\alpha \wedge \star\beta) = 0$ by Stokes' theorem, and from $d(\alpha \wedge \star\beta) = (d\alpha) \wedge \star\beta - \alpha \wedge \star\delta\beta$ we get $(d\alpha, \beta) = (\alpha, \delta\beta)$.

With this, we define the Laplace operator $\Delta : \Omega^k(M) \to \Omega^k(M)$ as $\Delta = d\delta + \delta d$. We define $\mathcal{H}^k(M) := \ker \Delta$ to be the space of *harmonic forms* on $M$. On compact manifolds $\Delta$ is self-adjoint with respect to the $L^2$ inner product. Additionally, we have

$$\begin{aligned}(\Delta\omega, \omega) &= (d\delta\omega, \omega) + (\delta d\omega, \omega) \\ &= (\delta\omega, \delta, \omega) + (d\omega, d\omega) \\ &= ||\delta\omega||^2 + ||d\omega||^2,\end{aligned}$$

which shows that $\omega$ is harmonic if and only if $d\omega = \delta\omega = 0$. Hence, any harmonic $k$-form is closed, and we have an inclusion $\mathcal{H}^k(M) \hookrightarrow Z^k(M)$. The classical Hodge theorem tells us that this inclusion induces an isomorphism $\mathcal{H}^k(M) \cong H^k_{dR}(M)$. A proof can be found in [21, Corollary 3.4.2]. More precisely, we have the following orthogonal decomposition with respect to the $L^2$-inner product on forms.

**Theorem 1.2.10** (Hodge decomposition)**.** *For each $0 \leq k \leq n$, the space of $k$-forms on $M$ decomposes as*

$$\Omega^k(M) = \mathcal{H}^k(M) \bigoplus d(\Omega^{k-1}(M)) \bigoplus \delta(\Omega^{k+1}(M)).$$

*In particular, any cohomology class has a unique harmonic representative.*

We may also realize Poincaré duality on closed manifolds using this result; the Hodge star operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ maps harmonic forms to harmonic forms, giving an isomorphism $\mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$.

## 1.3 The Hopf conjecture

**Conjecture 1.3.1** (Hopf conjecture)**.** *Let $M$ be a closed $2k$-manifold with nonpositive sectional curvature. Then*

$$(-1)^k \chi(M) \geq 0.$$

In view of the Cartan-Hadamard Theorem 1.2.7, this conjecture applies to all complete connected Riemannian manifolds of nonpositive curvature. In its original form, attributed to Hopf, the conjecture states that the Euler characteristic of closed Riemannian $2k$-manifolds with negative sectional curvature satisfies $(-1)^k \chi(M) > 0$, and for positive sectional curvature $\chi(M) > 0$. Later, it was generalized to nonpositive and nonnegative sectional curvature respectively, allowing for the Euler characteristic to vanish. The modern, generalized formulation of this conjecture, the Hopf-Thurston conjecture, is for aspherical spaces. By the Cartan-Hadamard theorem, manifolds of nonpositive sectional curvature are aspherical, and Thurston conjectured that $(-1)^k \chi(M) \geq 0$ holds for all closed aspherical $2k$-manifolds.

For a closed surface $S$, all the versions of the conjecture, including for positive sectional curvature, follow immediately from the Gauss-Bonnet theorem

$$\chi(S) = \frac{1}{2\pi} \int_S K dA,$$

where $K = \kappa_1 \cdot \kappa_2$ is the Gaussian curvature of $S$. Nonpositive curvature simply means that $K \leq 0$ as a function on $S$[1], and hence $(-1)^k\chi(S) = \frac{-1}{2\pi}\int K dA \geq 0$. The case of surfaces also explains the sign $(-1)^k$ in the Hopf conjecture; as the Euler characteristic is multiplicative under products, and nonpositive sectional curvature is conserved under products (with the product metric), any $2k$-manifold of the form $S_1 \times \ldots S_k$ for surfaces $S_i$ of nonpositive curvature will have Euler characteristic of sign $(-1)^k$ or 0.

The Chern-Gauss-Bonnet theorem, proved by Chern in [9], is a generalization of the Gauss-Bonnet theorem to all even-dimensional Riemannian manifolds. The theorem states that the Euler characteristic of a closed Riemannian $2k$-manifold $(M, g)$ for $k \geq 1$ is given by

$$\chi(M) = \frac{1}{(2\pi)^k}\int_M \mathrm{Pf}(\Omega),$$

where $\mathrm{Pf}(\Omega)$ is the *Pfaffian* of the *curvature form* $\Omega$ associated to the Levi-Civita connection of $g$. In [8], Chern proves (and attributes to Milnor) that for 4-manifolds of nonpositive sectional curvature, the Chern-Gauss-Bonnet integrand $\mathrm{Pf}(\Omega)$ is nonnegative as a function on $M$, proving the $k = 2$ case of the Hopf conjecture. Hence, the Hopf conjecture for smooth manifolds holds in $\leq 4$ dimensions.

**Theorem 1.3.2.** *[8, Theorem 5] If $M$ is a closed $2k$-manifold of nonpositive sectional curvature for $k \leq 2$, then*

$$(-1)^k\chi(M) \geq 0.$$

The approach to the Hopf conjectures using the Chern-Gauss-Bonnet theorem was known as the algebraic Hopf conjecture, which was disproved by Geroch in [16], where he constructs examples of curvature tensors with positive sectional curvature but Chern-Gauss-Bonnet integrand of the wrong sign.

## 1.4 $\ell^2$-cohomology and the Dodziuk-Singer conjecture

In this subsection we give an analytic definition of $\ell^2$-cohomology, based on [25, Section 1.3] and [6]. Throughout this subsection, $M$ is a smooth, complete manifold without boundary, which is not necessarily compact. For example, the universal covers of closed manifolds with the lifted metric are such spaces, which is the case we are most interested in.

We have an $L^2$-inner product and norm on the space of compactly supported smooth

[1] Note that for $\dim M > 2$, the sectional curvature $K$ is not a function on $M$, but on the 2-Grassmann bundle of $TM$.

$k$-forms $\Omega^k_c(M)$ defined by

$$(\alpha,\beta) := \int_M \alpha\wedge\star\beta = \int_M \langle\alpha,\beta\rangle_g dV$$
$$||\omega||_2 := \sqrt{(\omega,\omega)}.$$

Of course, when $M$ is compact, this gives an inner product and norm of $\Omega^k(M)$. We may extend the inner product to the whole of $\Omega^*(M)$ by setting $(\alpha,\beta)=0$ if $\alpha,\beta$ are not of the same degree. On non-compact manifolds $\int_M \alpha\wedge\star\beta$ may of course diverge for $k$-forms $\alpha,\beta$ which are not compactly supported.

**Definition 1.4.1.** A $k$-form $\omega\in\Omega^k(M)$ is an $\ell^2$ *$k$-form* if

$$||\omega||_2 = \int_M \omega\wedge\star\omega < \infty.$$

We define the space of $\ell^2$-forms on $M$, $\Omega^k_{(2)}(M)$, as the Hilbert space completion of $\Omega^k_c(M)$ with respect to the $L^2$-product. As a set, this coincides with the space of $k$-forms that are $\ell^2$. Define

$$Z^k_{(2)}(M) := \{\omega\in\Omega^k_{(2)}(M) \mid d\omega = 0\},$$

and

$$B^k_{(2)} := \{\omega\in\Omega^k_{(2)}(M) \mid \text{there exists } \eta \in \Omega^{k-1}_{(2)}(M) \text{ so that } d\eta=\omega\}.$$

The exterior derivative of an $\ell^2$-form is not necessarily $\ell^2$, which is why we restrict to those forms $\omega$ for which $d\omega$ is $\ell^2$.

**Definition 1.4.2.** The $k^{th}$ *(reduced) $\ell^2$-cohomology group* $H^k_{(2)}(M)$ of $M$ is defined as

$$H^k_{(2)}(M) = Z^k_{(2)}(M)/\overline{B^k_{(2)}},$$

where the closure is taken with respect to the $\ell^2$-topology induced by $||\cdot||_2$.

The image of $d$ need not be closed in $\Omega^k_{(2)}(M)$ when $d$ is restricted to those $\ell^2$-forms whose differential is $\ell^2$, which is why we take the closure. Not taking the closure results in the *unreduced* $\ell^2$-cohomology groups. When $M$ is complete, $\overline{B^k_{(2)}} = \overline{d(\Omega^{k-1}_c(M))}$, the closure of the image of compactly supported $(k-1)$-forms [6, Lemma 1.5].

Similarly to the compact case, for compactly supported forms $\alpha,\beta$, the exterior differential $d^k:\Omega^k(M)\to\Omega^{k+1}(M)$ has an adjoint with respect to $(\cdot,\cdot)$, given by $\delta^k = (-1)^{nk+n+1}\star$

$d^{n-k}\star : \Omega^k(M) \to \Omega^{k-1}(M)$, i.e., $(d\alpha, \beta) = (\alpha, \delta\beta)$. When $M$ is complete, this result extends to square-integrable forms $\omega, \eta$ if $d\omega$ and $\delta\eta$ are also square-integrable, which we will prove in Proposition 2.3.2. The completeness of $M$ is crucial, as it guarantees the existence of a sequence of bump functions (smooth functions with compact support) $f_n : M \to [0,1]$, which allow us to reduce to the case where $\omega, \eta$ have compact support.

As in the case of Hodge theory on compact manifolds, the Laplacian operator on $\Omega^*(M)$ plays a vital role. It is defined as usual by

$$\Delta = d\delta + \delta d \ : \Omega^*(M) \to \Omega^*(M).$$

The space of *$\ell^2$ harmonic $k$-forms* on $M$ is defined as

$$\mathcal{H}^k_{(2)}(M) := \{\omega \in \Omega^k(M) \mid \Delta\omega = 0,\ ||\omega||_2 < \infty\}.$$

For a closed manifold $M$, the Hodge and de Rham theorems tell us that the Betti numbers of $M$, defined as the ranks of the singular homology groups of $M$, can be computed as the dimensions of the spaces of harmonic differential forms of varying degree. For a complete manifold $M$ without boundary, we have the following $\ell^2$-version of the Hodge theorem, showing that the space of harmonic $\ell^2$-forms computes the (reduced) $\ell^2$-cohomology of $M$.

**Lemma 1.4.3** ([25, Lemma 1.72]). *Let $M$ be a complete Riemannian manifold without boundary. Then the inclusion of $\mathcal{H}^k_{(2)}(M)$ into $\Omega^k_{(2)}(M)$ induces an isomorphism*

$$\mathcal{H}^k_{(2)}(M) \xrightarrow{\sim} H^k_{(2)}(M).$$

Note that this is not necessarily the case for non-complete $M$.

As mentioned above, on compact manifolds the space of harmonic forms coincides with the intersection of the kernels of $d$ and $\delta$. This result holds whenever $M$ is complete.

**Proposition 1.4.4** ([29, Theorem 26]). *Let $M$ be a Riemannian manifold with $\mathcal{H}^k_{(2)}(M)$ the space of $\ell^2$-harmonic forms. If $M$ is complete, then*

$$(\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (\delta\alpha, \delta\alpha).$$

*In particular, we have*

$$\mathcal{H}^k_{(2)}(M) = \{\omega \in \Omega^k_{(2)}(M) \ \mid \ d\omega = \delta\omega = 0\}.$$

This is however not true in general; for example, on $M = [0,1] \subset \mathbb{R}$, the harmonic functions are affine, while solutions to $df = \delta f = 0$ are constant.

In the complete case, we have the following analogue of the Hodge decomposition for $\ell^2$-forms [25, Theorem 1.57].

**Theorem 1.4.5** ($\ell^2$-Hodge decomposition)**.** *Let $M$ be a complete manifold, and $\Omega^k_{(2)}(M)$ the space of $\ell^2$ $k$-forms on $M$. Then*

$$\Omega^k_{(2)}(M) = \mathcal{H}^k_{(2)} \bigoplus \overline{d(\Omega^{k-1}_{(2)}(M))} \bigoplus \overline{\delta(\Omega^{k+1}_{(2)}(M))}\ ,$$

*where $\overline{d(\Omega^{k-1}_{(2)}(M)}$ is the closure of the intersection of the image of $d$ and $\Omega^k_{(2)}(M)$, and similarly for $\overline{\delta(\ell^2\Omega^{k+1})}$.*

In general, the space of $\ell^2$-harmonic forms on $M$ is not a topological invariant of $M$, but depends on the quasi-isometry class of the metric.

**Proposition 1.4.6.** *The space $\mathcal{H}^*_{(2)}(M)$ of harmonic $\ell^2$-forms is a quasi-isometry invariant of $(M,g)$: if $g_1$ and $g_2$ are two complete metrics on $M$ such that there exists a $1 < \lambda < \infty$ with*

$$\lambda^{-1} g_1 \leq g_2 \leq \lambda g_1,$$

*then*

$$\mathcal{H}^*_{(2)}(M, g_1) \cong \mathcal{H}^*_{(2)}(M, g_2).$$

In fact, a more refined version of the invariance of $\mathcal{H}^*_{(2)}(M)$ holds. A continuous map $f : M \to N$ of Riemannnian manifolds is *Lipschitz* if $d_N(f(x), f(y)) \leq \lambda d_M(x,y)$ for some $\lambda > 0$, where $d_N, d_M$ are the distance functions on $N$ and $M$ respectively. For smooth $f$, this condition is equivalent to $||df||_\infty = \sup_{p \in M} |df(p)|_g$ being bounded by a constant. Now, if $f : M \to N$ is a bi-Lipschitz homeomorphism, meaning that both $f$ and $f^{-1}$ are Lipschitz homeomorphisms, then it induces an isomorphism $f^* : \mathcal{H}^*_{(2)}(N) \to \mathcal{H}^*_{(2)}(M)$ [18, 1.1.E.]. When $M$ and $N$ are compact, one recovers the usual homotopy invariance of $\mathcal{H}^*(M)$ and $\mathcal{H}^*(N)$, as all continuous maps can be approximated by Lipschitz maps.

The de Rham theorem for closed manifolds tells us that de Rham cohomology, or by the Hodge theorem the spaces of harmonic forms on a manifold, agree with the singular cohomology groups. In the $\ell^2$-case, we have a similar theorem relating analytic $\ell^2$-cohomology with a cellular version of $\ell^2$-invariants, whose precise definition we postpone to Section 6. The definition is especially suited to studying the universal cover $p : \widetilde{M} \to M$ of a closed manifold $M$, where $\pi_1(M)$ acts cocompactly on $\widetilde{M}$ by deck transformation.

**Theorem 1.4.7** ([25, Theorem 1.59]). *Let $M$ be a closed Riemannian manifold with a smooth triangulation, and lift the metric and triangulation to the universal cover $\widetilde{M}$ of $M$. Then, the integration of closed $k$-forms over $k$-simplices defines an isomorphism*

$$\mathcal{H}^k_{(2)}(\widetilde{M}) \xrightarrow{\sim} H^k_{(2)}(\widetilde{M}),$$

*where $\mathcal{H}^k_{(2)}(\widetilde{M})$ is the space of harmonic $\ell^2$ $k$-forms and $H^k_{(2)}(\widetilde{M})$ is the $k^{th}$ cellular $\ell^2$-cohomology group, computed using the lifted triangulation.*

When nonzero, the $\ell^2$-cohomology groups of noncompact manifolds tend to be infinite-dimensional. However, we can obtain more information in the case where we are computing the $\ell^2$-cohomology of a $G$-manifold $M$, where $G$ acts freely on $M$ such that the quotient $M \to M/G$ is a covering map, in particular when computing the $\ell^2$-cohomology of the universal cover $\widetilde{M}$ of some closed manifold $M$, where $\pi_1(M)$ acts on $\widetilde{M}$ freely such that the quotient $\widetilde{M} \to \widetilde{M}/\pi_1(M) \cong M$ is the universal covering map. In Section 6, we will show how to assign a real number $b^{(2)}_k$ to $H^k_{(2)}(M)$, called the $k^{th}$ *$\ell^2$-Betti number* of $\widetilde{M}$, which is the von Neumann dimension of the Hilbert $\mathcal{N}(\pi_1(M))$-module $H^k_{(2)}(\widetilde{M})$ (6.1.8).

For completeness, let us mention that this number can also be defined analytically; if $\mathcal{F}$ is a *fundamental domain* for the $G$-action on $M$, i.e., an open subset of $M$ such that $M = \cup_{g \in G} g.\overline{\mathcal{F}}$ and $g\mathcal{F} \cap \mathcal{F} = \varnothing$ for all $g \neq e \in G$, then

$$b^{(2)}_k(M) = \lim_{t \to \infty} \int_{\mathcal{F}} \mathrm{trace}_{\mathbb{C}}(e^{-t\Delta_k}(x,x))\mathrm{vol},$$

where $e^{-t\Delta_k}$ denotes the heat kernel on $\Omega^k(M)$ [25, pp. 52].

One of the main properties of the $\ell^2$-Betti numbers is *faithfulness*: $b^{(2)}_k = 0$ if and only if $H^k_{(2)}(M) = 0$. As the problems of interest relating to this topic in this thesis concern the vanishing of certain $\ell^2$-Betti numbers of universal covers, we do not require this exact analytic definition, as by faithfulness it suffices to check whether $H^k_{(2)}(M)$ (or equivalently the space of $\ell^2$-harmonic forms) vanishes.

In general, the individual $\ell^2$-Betti numbers of $\widetilde{M}$ are not related to the usual Betti numbers of a closed manifold $M$. However, we have a relation between the alternating sum of these numbers. In [4], Atiyah proved an $\ell^2$-version of the famous Atiyah-Singer index theorem for elliptic operators. This result states that the index of an elliptic operator on a closed manifold can be computed as the index of the lift of this operator to the universal cover. On closed manifolds, the index of the operator $d+\delta : \Omega_{\text{even}}(M) \to \Omega_{\text{odd}}(M)$, which is elliptic, equals the Euler characteristic of $M$, while the lift of this operator to $\widetilde{M}$ has the $\ell^2$-Euler characteristic $\chi_{(2)}(\widetilde{M}) = \sum_{k=0}^{n}(-1)^k b_k^{(2)}(\widetilde{M})$ as its index. Applying Atiyah's result to these operators gives the following.

**Lemma 1.4.8** (Atiyah's formula [4]). *Let $p : \widetilde{M} \to M$ be the universal cover of a closed $n$-manifold $M$. Then*

$$\chi(M) = \sum_{k=0}^{n}(-1)^k b_k^{(2)}(\widetilde{M}).$$

Motivated by using this result to approach the Hopf conjecture, as well as supporting computations, Dodziuk and Singer suggested the following vanishing theorem for $\ell^2$-cohomology of closed aspherical manifolds.

**Conjecture 1.4.9** (Dodziuk-Singer conjecture). *Let $\widetilde{M}$ be the universal cover of a closed, aspherical $n$-manifold $M$. Then, the $\ell^2$-Betti numbers of $\widetilde{M}$ satisfy*

$$b_k^{(2)}(\widetilde{M}) = 0 \quad \text{if } k \neq \frac{n}{2}.$$

In some texts, it is additionally conjectured that if $n = 2k$ and $M$ has strictly negative sectional curvature, then the middle $\ell^2$-Betti number is nonzero. If the Dodziuk-Singer conjecture holds, then by Atiyah's formula 1.4.8 we have $(-1)^k\chi(M) = b_k^{(2)}(\widetilde{M}) \geq 0$ for a closed $2k$-manifold $M$, and hence the Dodziuk-Singer conjecture implies the Hopf-Thurston conjecture.

The original form of the Dodziuk-Singer conjecture stated that for a simply connected complete Riemannian $n$-manifold with nonpositive sectional curvature, the spaces $\mathcal{H}^k_{(2)}(M)$ vanish for all $k \neq \frac{n}{2}$, with no reference to covering spaces. This was disproved by Anderson in [1], where he constructed examples of simply connected complete Riemannian manifolds with negative sectional curvature that have nonzero harmonic forms outside the middle degree. However, these examples do not admit a cocompact free proper action of a group $G \leq \mathrm{Isom}(M)$, so they do not give counterexamples to conjecture 1.4.9.

## 2 $\ell^2$-cohomology of Kähler hyperbolic manifolds

In this section we prove the Dodziuk-Singer conjecture for Kähler hyperbolic manifolds, following Gromov [18]. To do so, we first develop the necessary results on differential forms and Hodge theory on complete manifolds that are not necessarily compact.

### 2.1 Kähler geometry

A *complex n-manifold* is a topological manifold $X$ together with an atlas $(U_\alpha, \varphi_\alpha)$ of open sets $U_\alpha$ with $X = \cup_\alpha U_\alpha$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$, such that the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic. Two such atlases $(U_\alpha, \varphi_\alpha), (V_\beta, \phi_\beta)$ are equivalent if their union is a holomorphic atlas. For any point $p \in X$, we may consider a local holomorphic coordinate system $z = (z_1, \ldots, z_n)$ on some open neighbourhood $U \ni p$. If $z_i = x_i + iy_i$, then $T_{p,\mathbb{R}}X$ has local coordinates $\partial_{x_i}, \partial_{y_i}$ for $i = 1, \ldots, n$. This can be complexified to $T_\mathbb{C}X = T_\mathbb{R}X \otimes_\mathbb{R} \mathbb{C}$, for which we use the coordinates $\partial_{z_i} = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$ and $\partial_{\bar{z}_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$. Dual to this we have the 1-forms $dz_i := dx_i + idy_i$ and $d\bar{z}_i = dx_i - idy_i$.

Let $V$ be a real vector space. A linear map $J : V \to V$ with $J^2 = -I$ is called an *almost complex structure* on $V$. The operator $J$ extends to the complexification $V \otimes_\mathbb{R} \mathbb{C}$, and has eigenvalues $\pm i$. An almost complex structure on a Riemannian manifold $(M, g)$ is an almost complex structure on the tangent bundle $TM$. Any complex manifold $X$ admits a natural complex structure, locally given by $I : T_pX \to T_pX$, $\partial_{x_i} \mapsto \partial_{y_i}$, $\partial_{y_i} \mapsto -\partial_{x_i}$. A complex manifold $X$ with a Riemannian metric $g$ is *Hermitian* if $g$ is compatible with $J$, meaning that $g_p(v, w) = g_p(Jv, Jw)$ for all $p \in X$ and $v, w \in T_pX$.

**Definition 2.1.1.** Let $(X, g, J)$ be a Hermitian manifold with almost complex structure $J$, i.e., $g(X, Y) = g(JX, JY)$, and let $\omega$ be the form $\omega(X, Y) := g(JX, Y)$. Then $X$ is a *Kähler manifold* if $d\omega = 0$, and $\omega$ is the *Kähler form* of $X$.

Locally, the form $\omega$ is given by

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\bar{z}_j,$$

where $[h_{ij}]_{i,j=1}^n$ is a positive definite hermitian matrix. Combining the local theory of Hermitian vector spaces with Hodge theory allows to prove very strong results on the cohomology of compact Kähler manifolds, culminating in the Kähler package, which includes results such as the hard Lefschetz theorem, a refined version of the Hodge decomposition, and the Hodge-Riemann bilinear relations. As we do not require the full force of Hodge theory on compact Kähler manifolds, we refer to [20] for a more detailed introduction to the topic. Gromov's proof of the Dodziuk-Singer conjecture for Kähler hyperbolic manifolds relies on the analysis of the Lefschetz operators $\mathcal{L}^k : \Omega^p_{(2)}(X) \to \Omega^{p+2k}_{(2)}(X)$, $\mathcal{L} : \alpha \mapsto \omega^k \wedge \alpha$ on complete Kähler manifolds.

## 2.2 Complete manifolds and bounded forms

Let $(M, g)$ be a Riemannian $n$-manifold. With the inner product $g_p$ on $T_pM$, we get an inner product on $\bigwedge^k T_pM$ by $\langle \alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta \rangle = \det\left(g_p(\alpha_i, \beta_j)^k_{i,j=1}\right)$, and one step further an inner product on $\left(\bigwedge^k T_pM\right)^* \cong \bigwedge^k T_pM^*$. The dual norm on $\bigwedge^k T_pM^*$ is given by $|\alpha(p)| = \sup_{(v_1,\ldots,v_k)\in T^1_pM} |\alpha(v_1, \ldots, v_k)|$, where $T^1_pM$ denotes the unit sphere in $T_pM$ with respect to the metric induced by $g_p$. With this, the $L^\infty$-norm for a differential form $\alpha \in \Omega^*(M)$ is defined as

$$||\alpha||_\infty := \sup_{p\in M} |\alpha(p)|_g.$$

**Definition 2.2.1** (Bounded forms)**.** Let $M = (M, g)$ be a Riemannian manifold with metric $g$. A differential form $\alpha$ is *bounded* with respect to $g$ if the $L^\infty$-norm of $\alpha$ is finite, i.e.,

$$\|\alpha\|_{L^\infty} = \sup_{p\in M} |\alpha(p)|_g < \infty.$$

The form $\alpha$ is *d-bounded* if $\alpha = d\beta$ for a bounded form $\beta$.

For a differential form $\alpha$, we consider the pointwise norm $|\alpha| = |\alpha(x)|$ as a function on $M$. On compact manifolds, every smooth form is bounded, and hence $d$-bounded if and only if it is exact. However, if $M$ is not compact, this is not the case.

**Example 2.2.2.**

(i) The form $\alpha = dx_1 \wedge \ldots \wedge dx_n$ is clearly bounded and exact on $\mathbb{R}^n$ with the Lebesgue measure $\mu$, but not $d$-bounded: if $\alpha = d\beta$ for an $(n-1)$-form $\beta$, for ball $B = B(0, R)$ we have $\mu_n(B) = \int_B \alpha = \int_{\partial B} \beta \leq \|\beta\|_{L^\infty} \mu_{n-1}(\partial B)$, so that $\|\beta\|_{L^\infty} \geq \frac{\mu_n(B)}{\mu_{n-1}(\partial B)} = \frac{R}{n}$. Hence, taking $R \to \infty$ shows that $\|\beta\|_{L^\infty}$ is not bounded. One can similarly show the claim for any $dx^I$ for a multi-index $I$.

(ii) If $\alpha$ is a closed 1-form on $M$ with $\alpha = df$ for some smooth function $f$, then $|df| = |\alpha| \leq \|\alpha\|_{L^\infty}$. Hence, if $\alpha$ is bounded, $f$ is Lipschitz continuous with $|df|$ bounded by $\|\alpha\|_{L^\infty}$.

**Definition 2.2.3** ($\tilde{d}$-bounded forms)**.** Let $p : \widetilde{M} \to M$ be the universal cover of $M$, and $\tilde{\alpha}$ the lift of a form $\alpha$ on $M$ to $\widetilde{M}$. The form $\alpha$ is *$\tilde{d}$-bounded* if $\tilde{\alpha}$ is $d$-bounded on $\widetilde{M}$ with respect to the lift $p^*g = \tilde{g}$ of $g$ to $\widetilde{M}$.

With respect to the lifted metric, the projection $p : \widetilde{M} \to M$ becomes a local isometry, so $||\alpha||_{\infty,\widetilde{M}} = ||\alpha||_{\infty,M}$, and hence being $d$-bounded implies $\tilde{d}$-boundedness. However, the converse not true; a $\tilde{d}$-bounded form $\alpha$ need not even be exact on $M$, which can happen for example when $M$ is aspherical and topologically non-trivial, so that the de Rham cohomology of $M$ becomes an obstruction to finding a form $\beta$ with $d\beta = \alpha$. On compact manifolds $\tilde{d}$-boundedness does not depend on the metric, as all smooth metrics are equivalent. Furthermore, when $M$ is compact, the $\tilde{d}$-boundedness of a $k$-form $\alpha$ depends only on its cohomology class $[\alpha] \in H^k_{dR}(X, \mathbb{R})$: suppose $\alpha$ is $\tilde{d}$-bounded with $\tilde{\alpha} = d\beta$ for a bounded form $\beta \in \Omega^{k-1}(\widetilde{M})$, and consider $\alpha' = \alpha + d\tau$ for some $\tau \in \Omega^{k-1}(M)$. Then the lift $\tilde{\alpha}'$ equals $d(\beta + \tilde{\tau})$, and

$$||\beta + \tilde{\tau}||_{\infty,\widetilde{M}} \leq ||\beta||_{\infty,\widetilde{M}} + ||\tilde{\tau}||_{\infty,\widetilde{M}} = ||\beta||_{\infty,\widetilde{M}} + ||\tau||_{\infty,M} < \infty,$$

where the finiteness of $||\tau||_{\infty,M}$ follows from compactness.

In extending results from Hodge theory to the non-compact case, completeness is crucial, as it guarantees the existence of cutoff functions, which we use to reduce problems to the compact case.

**Definition 2.2.4** (Cutoff functions)**.** Let $M$ be a complete Riemannian manifold and $\epsilon > 0$. A *cutoff function* $a_\epsilon$ satisfies the following properties:

(i) $a_\epsilon$ is a smooth compactly supported function from $M$ to the interval $[0, 1]$.

(ii) The set $\{x \in M \mid a_\epsilon(x) = 1\}$ exhausts $M$ as $\epsilon \to 0$.

(iii) The differential of $a_\epsilon$ is everywhere bounded by $\epsilon$, i.e.,

$$\|da_\epsilon\|_{L_\infty} := \sup_{x\in M} |da_\epsilon(x)| \leq \epsilon.$$

### 2.3 $\ell^2$-Hodge theory on complete manifolds

The standard tools of Hodge theory on compact Kähler manifolds, including the various differentials $d, \delta, \partial, \bar{\partial}, \Delta := d\delta + \delta d$, the Hodge star operator $\star$, and the Lefschetz operator $\mathcal{L}$, are defined locally, and do not require the manifold to be closed. However, the fact that $M$ is closed is necessary for deriving some important results. For example, when showing that $(d\alpha, \beta) = (\alpha, \delta\beta)$ with respect to the $L^2$-inner product, we use the fact that $M$ has no boundary when using Stokes' formula, which tells us that $\int_M d(\alpha \wedge \star\beta) = \int_{\partial M} \alpha \wedge \star\beta = 0$. The same holds for compactly supported forms on non-compact manifolds without boundary. This is however no longer necessarily the case in general when $M$ is not closed, as we have to take the boundary terms $\int_M d\alpha = \int_{\partial M} \alpha$ into account. However, when $M$ is complete with no boundary and $\alpha$ is an $L^1$-form on $M$, the vanishing of $\int_M d\alpha$ still holds:

**Lemma 2.3.1.** *Let $\eta$ be an $L^1$-form on $M$ of degree $n-1$, i.e., $\|\eta\|_{L_1} = \int_M |\eta(x)| vol < \infty$, such that $d\eta$ is also $L^1$. If $M$ is complete, then $\int_M d\eta = 0$.*

*Proof.* Consider the form $a_\epsilon \eta$, which has compact support by the properties of the cutoff function $a_\epsilon$. By Stokes' formula 1.2.8 for compactly supported forms $0 = \int_M d(a_\epsilon \eta) = \int_M da_\epsilon \eta \wedge \eta + \int_M a_\epsilon d\eta$, and as $|\int_M a_\epsilon d\eta| \leq |\int_M da_\epsilon \wedge \eta| \leq \epsilon \|\eta\|_{L_1}$ with $d\eta$ being $L^1$ by assumption, we get

$$\int_M d\eta = \lim_{\epsilon \to 0} \int_M a_\epsilon d\eta = 0,$$

where we have applied Lebesgue's dominated convergence for the cutoff functions $a_\epsilon \to 1$. □

**Proposition 2.3.2.** *Let $M$ be a complete Riemannian manifold, and $\alpha \in \Omega^k_{(2)}, \beta \in \Omega^{k+1}_{(2)}$ such that $d\alpha$ and $d\beta$ are $\ell^2$-forms. Then*

$$(d\alpha, \beta) = (\alpha, \delta\beta).$$

*Proof.* Let $a_n$ be a sequence of cutoff functions such that $|da_n| \leq \frac{\lambda}{n}$ as a function on $M$ for some $\lambda > 0$ and $a_n(p) \to 1$ as $n \to \infty$ for all $p \in M$. Then

$$\langle d(a_n \alpha), \beta \rangle = \langle da_n \wedge \alpha + a_n d\alpha, \beta \rangle. \tag{1.1}$$

Consider first the left hand side of (1.1). As $a_n \alpha$ is compactly supported, by Stokes' formula 1.2.8 $(d(a_n\alpha),\beta) = (a_n\alpha,\delta\beta)$, i.e.,

$$\int_M \langle d(a_n\alpha),\beta\rangle \mathrm{vol} = \int_M \langle a_n\alpha,\delta\beta\rangle \mathrm{vol},$$

and by Lebesgue dominated convergence

$$\lim_{n\to\infty}\int_M \langle a_n\alpha,\delta\beta\rangle \mathrm{vol} = (\alpha,\delta\beta). \tag{1.2}$$

On the right hand side of (1.1), using Cauchy-Schwartz and $|da_n| \leq \frac{\lambda}{n}$ we have the pointwise estimate

$$\langle da_n \wedge \alpha,\beta\rangle \leq |da_n||\alpha||\beta| \leq \frac{\lambda}{n}|\alpha||\beta|, \tag{1.3}$$

which goes to 0 as $n \to \infty$. Using Lebesgue convergence and (1.3) we get

$$\begin{aligned}
\lim_{n\to\infty}\int_M \langle da_n\wedge\alpha + a_n d\alpha,\delta\beta\rangle \mathrm{vol} &= \lim_{n\to\infty}\int_M \langle da_n\wedge\alpha,\delta\beta\rangle \mathrm{vol} + \lim_{n\to\infty}\int_M \langle a_n d\alpha,\delta\beta\rangle \mathrm{vol} \\
&= \lim_{n\to\infty}\int_M \langle a_n d\alpha,\delta\beta\rangle \mathrm{vol} \\
&= \int_M \langle d\alpha,\beta\rangle \mathrm{vol} = (d\alpha,\beta),
\end{aligned}$$

and comparing with (1.2) proves the claim. □

Hence, we see that if $\alpha, d\alpha$ and $\delta\alpha$ are all $\ell^2$-forms, then

$$(\Delta\alpha,\alpha) = (d\alpha,d\alpha) + (\delta\alpha,\delta\alpha)$$

for the Laplacian $\Delta = d\delta + \delta d$, which shows that any such harmonic form $\alpha$ satisfies $d\alpha = \delta\alpha = 0$. As mentioned in Proposition 1.4.4, this result in fact extends to any harmonic $\ell^2$-form $\alpha$ on a complete manifold without the additional $\ell^2$-requirement on $d\alpha, \delta\alpha$, so that $\alpha$ is $\ell^2$-harmonic if and only if $\alpha$ is $\ell^2$ and $d\alpha = \delta\alpha = 0$. We refer to [29, Theorem 26] for a proof in this generality.

As in the case of the Hodge decomposition for compact manifolds 1.2.10, using Lemma 2.3.1 and Proposition 1.4.4 one may prove the $\ell^2$-Hodge decomposition 1.4.5 for $\ell^2$-forms, which states that

$$\Omega^k_{(2)}(M) = \mathcal{H}^k_{(2)} \bigoplus \overline{d(\Omega^{k-1}_{(2)}(M))} \bigoplus \overline{\delta(\Omega^{k+1}_{(2)}(M))}.$$

For us, the main consequence of this decomposition is the following:

**Corollary 2.3.3.** *If $\alpha$ is an $\ell^2$-harmonic form such that $\alpha = d\beta$ for some $\ell^2$-form $\beta$, then $\alpha = 0$.*

## 2.4 Vanishing theorem for Kähler hyperbolic manifolds

**Definition 2.4.1** (Kähler hyperbolic manifolds)**.** A closed complex manifold $X$ is *Kähler hyperbolic* if it admits a Kähler form $\omega$ which is $\tilde{d}$-bounded.

Examples of Kähler hyperbolic manifolds include ([18, p. 265]):

- Closed Kähler manifolds that are homotopy equivalent to a Riemannian manifold with strictly negative sectional curvature;
- Closed Kähler manifolds $X$ such that $\pi_2(X) = 0$ and the fundamental group of $X$ is *word-hyperbolic*, as in [17];
- Closed manifolds whose universal cover is a symmetric Hermitian space of noncompact type;
- Complex submanifolds and products of Kähler hyperbolic manifolds with the induced metrics.

Let $X$ be a Kähler manifold of dimension $n = 2m$ and $\omega$ the Kähler form of $X$. The form $\omega$ is a closed nonsingular 2-form which is parallel to the Levi-Civita connection $\nabla$ on $X$, i.e., $\nabla\omega = 0$. The first condition implies that the top power $\omega^m = \omega \wedge \ldots \wedge \omega$ does not vanish on $X$. Define the *Lefschetz operator* $\mathcal{L}^k : \Omega^p(M) \to \Omega^{p+2k}(X)$ by $\mathcal{L}^k(\eta) = \omega^k \wedge \eta$ for all $p$-forms $\eta$ and $k \in \mathbb{N}$. On complete Kähler manifolds, the Lefschetz opertor satisfies various desirable properties. Firstly, the Lefschetz operator $\mathcal{L}^k$ commutes with both $d$ and $\Delta$, and thus $\mathcal{L}^k$ sends harmonic forms to harmonic forms [20, Chapter 3.1]. In addition, in the non-compact case, the operator $\mathcal{L}^k : \Omega^p(M) \to \Omega^{p+2k}(X)$ restricts to a bounded operator on square-integrable forms, giving well-defined maps $\mathcal{L}^k : \Omega^p(M)_{(2)} \to \Omega^{p+2k}_{(2)}(X)$ and $\mathcal{L}^k : \mathcal{H}^p_{(2)}(X) \to \mathcal{H}^{p+2k}_{(2)}(X)$ [25, Theorem 11.28. (2)]. The analogous properties hold for the adjoint operator $\Lambda_k := (\mathcal{L}^k)^* : \Omega^{p+2k}(X) \to \Omega^p(M)$, again taken with respect to $L^2$-inner product on forms.

Next, we want to show that if the Kähler form $\omega$ is $\tilde{d}$-bounded, then $\mathcal{H}^p_{(2)}(X) = 0$ for any $p \neq m = \frac{1}{2}\dim X$. The following result is a standard result in local Kähler theory, and is proved in for instance [20, Proposition 1.2.30]. We denote the local Lefschetz operator by $L^k$ and the local adjoint by $\lambda_k$.

**Lemma 2.4.2.** *Let $V$ be a $k$-vector space of dimension $n = 2m$, $\omega$ an exterior 2-form and $L^k : \bigwedge^p V \to \bigwedge^{p+2k} V$ the Lefschets operator of $\omega$ as before. If $\omega$ is nonsingular, then $L^k$ is injective for $2p + 2k \leq n$ and surjective for $2p + 2k \geq n$.*

**Corollary 2.4.3.** *Let $\lambda_k : \bigwedge^{p+2k} V \to \bigwedge^p V$ be the adjoint operator to $L^k$. Then $\lambda_k$ is injective for $2p + 2k \leq n$.*

*Proof.* Fix a nonzero $\beta \in \bigwedge^{p+2k} V$ and let take any nonzero $\alpha \in \bigwedge^p V$. Suppose $\lambda_k \beta = 0$. By adjointness $\langle L^k \alpha, \beta \rangle = \langle \alpha, \lambda_k \beta \rangle = 0$ (with the induced inner product on the exterior power from that of $V$). As $L^k$ is injective, $L^k \alpha \neq 0$, and as $\alpha$ is arbitrary, by non-degeneracy we get that $\beta = 0$. □

**Theorem 2.4.4** ($\ell^2$-Hard Lefschetz theorem)**.** *The map $\mathcal{L}^k : \mathcal{H}^p_{(2)}(X) \to \mathcal{H}^{p+2k}_{(2)}(X)$ is injective on harmonic forms for $2p + 2k \leq n = \dim_{\mathbb{R}}(X)$ and surjective for $2p + 2k \geq n$.*

*Proof.* Injectivity follows from Lemma 2.4.2; if $\alpha \in \Omega^p(X)$ and $2p + 2k \leq n$, then $\mathcal{L}^k(\alpha) = \omega^k \wedge \alpha$ is zero if and only if it is zero locally. For surjectivity, by Corollary 2.4.3 the operator $\Lambda_k$ adjoint to $\mathcal{L}^k$ is also injective, and hence $\mathcal{L}^k$ has dense image, as a linear operator is injective if and only if the image of the adjoint is dense. As $\mathcal{L}^{k+l} = \mathcal{L}^k \circ \mathcal{L}^l$, to prove surjectivity it suffices to show the claim for $2p + 2k = n$, where $\mathcal{L}^k$ is injective. Now, injectivity shows that $\mathcal{L}^k$ is quasi-isometry, i.e., we have

$$\lambda^{-1} ||\omega|| \leq ||L^k \omega|| \leq \lambda ||\omega||$$

for some positive constant $\lambda < \infty$. The image of a quasi-isometric linear operator is closed, which along with the density of the image shows that $L^k$ is surjective for $2p + 2k = n$, and hence surjective for $2p + 2k \geq n$. □

**Theorem 2.4.5** ($\ell^2$-vanishing theorem)**.** *Suppose $(X, \omega)$ is a Kähler manifold where the Kähler form $\omega$ is $d$-bounded. Then $\mathcal{H}^p_{(2)}(\widetilde{X}) = 0$ for $p \neq m = \frac{n}{2}$.*

*Proof.* Let $\omega = d\eta$ for a bounded 1-form $\eta$. For any closed $\ell^2$-form $\alpha$, define $\beta := \eta \wedge \omega^{k-1} \wedge \alpha$.

Then $\mathcal{L}^k(\alpha) = \omega^k \wedge \alpha$ is closed and exact as

$$\begin{aligned} d\beta &= d(\eta \wedge \omega^{k-1} \wedge \alpha) \\ &= d\eta \wedge \omega^{k-1} \wedge \alpha \pm \eta \wedge d(\omega^{k-1} \wedge \alpha) \\ &= \omega \wedge \omega^{k-1} \wedge \alpha \\ &= \omega^k \wedge \alpha. \end{aligned}$$

Note that $\beta$ is $\ell^2$, as $\alpha$ is $\ell^2$ and $\eta \wedge \omega^{k-1}$ is bounded. If $\alpha$ is harmonic, then by Corollary 2.3.3 of the $\ell^2$-Hodge decomposition, $\mathcal{L}^k\alpha = 0$ for $k > 0$, as $\mathcal{L}^k$ preserves harmonicity, which by the injectivity of $\mathcal{L}^k$ implies $\alpha = 0$. □

As explained in Section 1, by Atiyah's Formula $\chi(X) = \sum_{p\geq 0}(-1)^p b_p^{(2)}(\widetilde{X})$ the Hopf conjecture for Kähler hyperbolic $2k$-manifolds follows, as $(-1)^k\chi(X) = \dim_\Gamma \mathcal{H}_2^k \geq 0$. The bulk of Gromov's paper [18] is dedicated to proving the nonvanishing of $\mathcal{H}_{(2)}^k(X)$, which combined with the above results yields the strict inequality

$$(-1)^k\chi(X) = \dim_\Gamma \mathcal{H}_{(2)}^k(\widetilde{X}) > 0.$$

**Remark 2.4.6.** Following the remarks on $\tilde{d}$-boundedness and curvature at the beginning of this section, we get that any compact Kähler $2k$-manifold $M$ with strictly negative sectional curvature satisfies $(-1)^k\chi(X) > 0$. However, we do not get the desired inequality for such spaces with $K(X) \leq 0$, for which we want to show that $(-1)^k\chi(X) \geq 0$; for example, a torus $\mathbb{T}^{2k} = \mathbb{C}^{2k}/\Lambda$ with the metric inherited from the Euclidean metric on $\mathbb{C}^{2k}$ is a compact Kähler manifold satisfying $\chi(X) = 0$. In [23], Jost and Zuo prove the vanishing theorem $\mathcal{H}_{(2)}^i = 0$ for $i \neq k$ for a larger class of spaces, known as Kähler nonelliptic manifolds, which contains both Kähler hyperbolic spaces as well as compact Kähler manifolds with nonpositive sectional curvature.

# 3 $\ell^2$-cohomology of rotationally symmetric manifolds

In this section we present Dodziuk's proof [13] of the Dodziuk-Singer conjecture for rotationally symmetric manifolds. As a consequence, we prove the Dodziuk-Singer conjecture for closed hyperbolic manifolds and closed manifolds covered by Euclidean space.

## 3.1 Rotationally symmetric manifolds

**Definition 3.1.1.** Let $M = (M, g)$ be a Riemannian manifold. $M$ is *rotationally symmetric at* $p \in M$ if the isotropy subgroup at $p$ of the isometry group $\mathrm{Isom}(M)$, i.e. $\mathrm{Isom}(M)_p = \{g \in \mathrm{Isom}(M) \mid g(p) = p\}$, is the orthogonal group $O(n)$.

Depending on whether $M$ is compact or not, the simply connected rotationally symmetric manifolds are topologically very simple: they are diffeomorphic to either $\mathbb{S}^n$ or $\mathbb{R}^n$. Here, we consider non-compact simply connected manifolds, in which case $M$ is rotationally symmetric if

(i) there exists some $p \in M$ such that the exponential mapping $\exp : T_pM \to M$ is a diffeomorphism, and

(ii) every linear isometry $\varphi : T_pM \to T_pM$ is the differential of an isometry $\Phi \in \mathrm{Isom}(M)$, i.e., $\Phi(p) = p$ and $d\Phi_p = \varphi$.

Hence $M$ is complete and can be identified with the tangent space $T_pM$ at $p$ via the exponential mapping. The point $p \in M$ is called a *pole*. In terms of geodesic polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}^{n-1} \cong M \setminus \{p\}$ we may write the Riemannian metric $ds^2$ as

$$ds^2 = dr^2 + f(r)^2 d\theta^2,$$

where $f(r) \in C^\infty([0, \infty))$ and satisfies $f(0) = 0$, $f'(0) = 1$ and $f(r) > 0$ for $r > 0$ [31, pp. 179-183], and $d\theta^2$ is the usual metric on $S^{n-1}$ induced by the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$.

## 3.2 Vanishing theorem for rotationally symmetric manifolds

As shown by Dodziuk, for this class of manifolds a complete description of the space $\mathcal{H}^*(M)$ of $\ell^2$-harmonic forms is given by the following theorem.

**Theorem 3.2.1.** *Let $M$ be a rotationally symmetric manifolds of dimension $n \geq 2$. Then* *(i)* $\mathcal{H}^p_{(2)}(M) = 0$ *for* $p \notin \{0, \frac{n}{2}, n\}$,

*(ii)* $\mathcal{H}^0_{(2)}(M) = \mathcal{H}^n_{(2)}(M) = \begin{cases} \mathbb{R}, & \text{if } \int_0^\infty f(r)^{n-1} dr < \infty, \\ 0 & \text{else,} \end{cases}$

*(iii) and if* $n = 2k$, $\mathcal{H}^k_{(2)}(M) = 0$ *if* $\int_1^\infty \frac{ds}{f(s)} = \infty$ *and* $\mathcal{H}^k_{(2)}(M)$ *is an infinite-dimensional Hilbert space if* $\int_1^\infty \frac{ds}{f(s)} < \infty$.

*Proof.* Let $V_0$ be the volume element for $S^{n-1}$, so that $dV = f(r)^{n-1} dV_0 \wedge dr$ is the volume element of $M$. Recall that as $M$ is complete, by Proposition 1.4.4 an $\ell^2$-form $\omega$ is harmonic if and only if $d\omega = \delta\omega = 0$. For (ii), if a square-integrable function $f$ is harmonic on $M$, then $df = 0$ implies $f$ is constant. Note that the integral $\int_0^\infty f(r)^{n-1} dr$ is a multiple of the volume of $M$, and constants are square-integrable and only if the total volume of $M$ is finite, which shows part (ii) after using the Hodge star isomorphism between $\mathcal{H}_0$ and $\mathcal{H}^n$.

Next, let us rewrite the conditions for harmonicity of an $\ell^2$ $k$-form $\omega$ in polar coordinates $(r, \theta)$. We may express a $k$-form $\omega$ on $M \setminus p$ as

$$\omega = a(r,\theta) \wedge dr + b(r,\theta)$$

where $a(r,\theta), b(r,\theta)$ are $(k-1)$ and $k$-forms respectively on $\mathbb{S}^{n-1}$. Formally, given any $k$-form $\omega$, the form $a$ is given by $a = (-1)^{k-1}\iota(\frac{\partial}{\partial r})\omega$, where $\iota$ denotes the interior product of $\omega$ and $\frac{\partial}{\partial r}$ (given by fixing the first input of $\omega$ as $\frac{\partial}{\partial r}$). Then, $b(r,\theta) = \omega - a(r,\theta) \wedge dr$.

Let $\star_0$ and $d_0$ denotes the Hodge star operator and exterior derivative on $\mathbb{S}^{n-1}$. For two conformal metrics $g \sim \lambda^2 g$ on an $n$-manifold, a basic fact about the respective Hodge star operators is that they are related by $\star_{\lambda^2 g} = \lambda^{n-2k} \star_g$. Then we get

$$\star\omega = (-1)^{n-k} f^{n-2k+1} \star_0 a(r,\theta) + f^{n-2k-1} \star_0 b \wedge dr.$$

Substituting this expression into the conditions for harmonicity, we get the following conditions.

$$\int_0^\infty \int_{\mathbb{S}^{n-1}} (f^{n-2k+1}|a|_0^2 + f^{n-2k-1}|b|_0^2) dV_0 dr < \infty \tag{1.4}$$

$$d_0 \star_0 a = d_0 b = 0 \tag{1.5}$$

$$d_0 a + (-1)^k \frac{\partial b}{\partial r} = 0 \tag{1.6}$$

$$\frac{\partial}{\partial r}(f^{n-2k+1} \star_0 a) + f^{n-2k-1} d_0 \star_0 b = 0 \tag{1.7}$$

As $\omega$ is a smooth $k$-form on the whole of $M$, we also require the pointwise norm $|\omega|^2 = f^{-2(k-1)}|a|_0^2 + f^{-2k}|b|_0^2$ to be bounded near $o \in M$, i.e., near $r = 0$, so that there exists some $C < \infty$ such that $|\omega|^2 < C$ for $r \in (0,1)$. Here $|\cdot|_0$ denotes the norm on $\mathbb{S}^{n-1}$. Note that as $\star_0, d_0$ are operators on $\mathbb{S}^{n-1}$, they commute with $\frac{\partial}{\partial r}$. Applying $\star_0$ to (1.7) and recalling the definition of the codifferential $\delta_0 = (-1)^k \star_0 d_0 \star_0$ on $k$-forms, the equation becomes

$$\frac{\partial}{\partial r}(f^{n-2k+1}a) + (-1)^k f^{n-2k-1}\delta_0 b = 0.$$

Note that if $\omega$ is harmonic, then $b = 0$ implies $a = 0$: by the equations (1.5), $a(r,\theta)$ is then harmonic on $\mathbb{S}^{n-1}$ for every $r > 0$, as $d_0 a = d_0 \star_0 a = 0$, with the second equality implying $\delta_0 a = 0$. However, $a(r,\theta)$ can be nonzero only when $\deg a = k - 1 = 0$, and by (1.7), we have $\frac{\partial}{\partial r}(f^{n-1}a) = 0$, so $a = \frac{C'}{f^{n-1}}$. This contradicts the boundedness of $|\omega|^2$ near $r = 0$ unless $C' = 0$, in which case $a = 0$.

Next, we want to cancel the term $\frac{\partial}{\partial r}(f^{n-2k+1)} \star_0 a)$ from (1.7); to do so, apply $d_0$ to (1.7) and use the commutativity with $\frac{\partial}{\partial r}$, and similarly multiply equation (1.6) by $-f^{n-2k+1}$ and apply $\frac{\partial}{\partial r}$ to get equal terms which contain derivatives of the form $a$. Adding this modified equation to $d_0$ of (1.7), we are left with

$$f^{n-2k-1}d_0\delta_0 b = \frac{\partial}{\partial r}(f^{n-2k+1}\frac{\partial b}{\partial r}).$$

Taking the $L^2$-inner product $(\cdot,\cdot)_0$ on $\Omega^k(\mathbb{S}^{n-1})$ with $b(\theta, r)$ and using the adjointness of $\delta_0$ and $d_0$ gives

$$\left(\frac{\partial}{\partial r}(f^{n-2k+1}\frac{\partial b}{\partial r}), b\right)_0 = f^{n-2k-1}(\delta_0 b, \delta_0 b)_0 \geq 0. \tag{1.8}$$

The derivative with respect to $r$ of the inner product is given by

$$\frac{d}{dr}\left(f^{n-2k+1}\frac{\partial b}{\partial r}, b\right)_0 = \left(\frac{\partial}{\partial r}(f^{n-2k+1}\frac{\partial b}{\partial r}), b\right)_0 + f^{n-2k+1}\left(\frac{\partial b}{\partial r}, \frac{\partial b}{\partial r}\right)_0, \tag{1.9}$$

which is nonnegative by (1.8), the nonnegativity of $f$ and $\left(\frac{\partial b}{\partial r}, \frac{\partial b}{\partial r}\right)_0 \geq 0$. As $|\omega|^2 = f^{-2(k-1)}|a|_0^2 + f^{-2k}|b|_0^2 < C$ near $r = 0$, we have $|b|_0^2 = O(r^{2k})$ for small $r$, as $f'(0) = O(1)$. Thus comparing exponents in (1.9), we see that $\left(f^{n-2k+1}\frac{\partial b}{\partial r}, b\right)_0 = O(r^n)$, so that

$$\frac{d}{dr}(b,b)_0 = 2\left(\frac{\partial b}{\partial r}, b\right)_0 \geq 0$$

for all $r > 0$. As $(b,b)_0 = ||b||_0^2$, the norm of $b$ is hence monotone and nondecreasing in $r$.

If $b \neq 0$, the norm of $b$ on the whole of $M \setminus p$ is given by

$$\int_0^\infty f^{n-2k-1}||b||_0^2 dr = ||b||^2 \leq ||\omega||^2,$$

and thus $\int_1^\infty f^{n-2k-1}dr < \infty$ if $\omega$ is a nonzero $\ell^2$-harmonic $k$-form for $k \neq 0, n$, and moreover by the Hodge star isomorphism we have $\mathcal{H}^k \cong \mathcal{H}^{n-k} \neq 0$. Thus, for these two dual spaces to be nonzero, we require both $\int_1^\infty f^{n-2k-1}dr$ and $\int_1^\infty f^{-n+2k-1}dr$ to be finite. If $k \neq \frac{n}{2}$ (in which case the integrals are equal), then either one of the exponents of $f$ in the integral is zero or have opposite signs, and in both cases one of the integrals has to diverge, which shows that $\mathcal{H}^k(M) = 0$ for $k \neq 0, n, \frac{n}{2}$. From this we also see that if $\int_1^\infty \frac{ds}{f(s)}$ diverges, then we have no harmonic forms in the middle degree, as $f^{n-2k-1} = f^{-1}$ for $k = \frac{n}{2}$.

Finally, we sketch the proof of the case of harmonic $\ell^2$-forms in the middle dimension when $\int_1^r \frac{ds}{f(s)} < \infty$. This requires the following result, due to Milnor:

**Lemma 3.2.2.** *Let $M$ be a rotationally symmetric manifold with metric $ds^2 = dr^2 + f(r)^2 d\theta^2$, and define*

$$R(r) = \exp\left(\int_1^r \frac{ds}{f(s)}\right).$$

*Then the mapping $F : M \setminus \{p\} \to \mathbb{R}^n \setminus \{0\}$ given by $F(r, \theta) = (R(r), \theta)$, in terms of geodesic polar coordinates on $M$ and polar coordinates of $\mathbb{R}^n$, extends to a $C^1$ conformal diffeomorphism onto an open ball of possibly infinite radius $\int_1^\infty \frac{ds}{f(s)}$. The map $F$ is smooth on $M \setminus \{p\}$.*

As the Hodge star operator depends only on the conformal structure on $M$ (up to scaling the metric), the conditions $d\omega = \delta\omega = 0$ and $\int_M \omega \wedge \star\omega < \infty$ are conformally invariant. Let $B$ be the open ball in $\mathbb{R}^n$ of radius $\int_1^\infty \frac{ds}{f(s)}$ centered at the origin.

Recall that the space of harmonic functions on open subsets on $\mathbb{R}^n$ for $n \geq 2$ is infinite dimensional; to see this, on an open subset of $\mathbb{R}^2$, the real part of any analytic function is harmonic. Representing such a function as a power series with infinite radius of convergence, we can embed the space of bounded sequences of complex numbers $(\lambda_j)_{j=1}^\infty$ into the space of harmonic functions over $\mathbb{C}$ by $(\lambda_j)_{j=1}^\infty \mapsto \mathrm{Re}(\sum_{j=1}^\infty j^{-j}\lambda_j z^n)$. As this space is obviously infinite-dimensional, we see that the space of harmonic functions on $\mathbb{R}^2$ is as well. Extending such a function $u$ to $\mathbb{R}^n$ for $n > 2$ by $u(x_1, \ldots, x_n) = u(x_1, x_2)$ of course preserves harmonicity, which shows the result for all $n \geq 2$. From this, we see that the space of smooth $k$-forms on $\mathbb{R}^n$ which satisfy $d\omega = \delta\omega = 0$ is infinite dimensional; for example, if $f : \mathbb{R}^{k+1} \to \mathbb{R}$ is harmonic, then $\omega = d(f(x_k, \dots, x_n)dx_1 \wedge \dots \wedge dx_{k-1}))$ satisfies $d\omega = 0$ by $d \circ d = 0$, and $\delta\omega = 0$ by the harmonicity of $f$, which shows that the space of harmonic $k$-forms on $\mathbb{R}^n$ has infinite dimension.

Now, when $\int_1^\infty \frac{ds}{f(s)} < \infty$, then the restriction of these harmonic forms to $B$, denoted by $\mathcal{H}_B^k$, are of course $\ell^2$, and by the above lemma and conformal invariance, the pullbacks $F^*\omega$ of such forms are $\ell^2$ and smooth on $M \setminus \{p\}$, and satisfy $dF^*\omega = \delta F^*\delta = 0$. Now, one must argue by elliptic regularity theory applied to the elliptic operator $(d + \delta)$ that these forms are smooth and harmonic on the whole of $M$, which proves that the pullback induces an isomorphism between $\mathcal{H}_B^k$ and $\mathcal{H}_{(2)}^k(M)$, so that $\mathcal{H}_{(2)}^k(M)$ is also infinite-dimensional. □

**Corollary 3.2.3.** *The $\ell^2$-cohomology of $\mathbb{R}^n$ with the Euclidean metric vanishes in all degrees. In particular, if $M$ is a closed Riemannian $n$-manifold whose universal cover $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$, with the lifted metric equivalent to the Euclidean metric, then*

$$\mathcal{H}_{(2)}^k(\widetilde{M}) = 0$$

*for all $k$.*

*Proof.* Euclidean space is clearly rotationally symmetric, with any point serving as a pole (in fact, whenever $M$ is rotationally symmetric and $\int_1^\infty \frac{ds}{f(s)} = \infty$, then $M$ is conformally equivalent to $\mathbb{R}^n$ with the Euclidean metric). We have $f(r) = r$ for the Euclidean metric $ds^2 = dr^2 + f(r)^2 d\theta^2$. Hence $\int_0^\infty f(r)^{n-1} dr$ and $\int_1^\infty \frac{ds}{f(s)}$ both diverge, and by Theorem 3.2.1 the $\ell^2$-cohomology of $\mathbb{R}^n$ vanishes also in the middle, top and bottom degrees. Thus, by Proposition 1.4.6 we get a proof of the Dodziuk-Singer conjecture for closed manifolds whose universal cover is $\mathbb{R}^n$, if the lifted metric is equivalent to the Euclidean metric. □

Of course, by Atiyah's formula we also get a proof of the Hopf conjecture for such manifolds. Similarly, we get the a proof of the conjectures for closed hyperbolic manifolds.

**Theorem 3.2.4.** *The $\ell^2$-cohomology $\mathcal{H}_{(2)}^k(\mathbb{H}^n)$ of hyperbolic space $\mathbb{H}^n$ satisfies*

$$\mathcal{H}_{(2)}^k(\widetilde{M}) \begin{cases} = 0 & \textit{if} \quad k \neq \frac{n}{2} \\ \neq 0 & \textit{for} \quad k = \frac{n}{2}, \end{cases}$$

*In particular, the Dodziuk-Singer conjecture holds for closed hyperbolic manifolds.*

*Proof.* In spherical coordinates, the Riemannian metric on $\mathbb{H}^n$ is given by $dr^2 + \sinh^2(r) d\theta^2$, so the result follows from applying Theorem 3.2.1 with $f(r) = \sinh(r)$. As $\int_1^\infty \frac{ds}{f(s)} < \infty$, we also get that the middle-dimensional $\ell^2$-Betti number is nonzero.

A standard fact in hyperbolic geometry states that every complete, connected and simply connected $n$-manifold of constant sectional curvature $-1$ is isometric to $\mathbb{H}^n$, and hence the universal cover of any hyperbolic closed Riemannian $n$-manifold, with the lifted metric, is isometrically diffeomorphic to $\mathbb{H}^n$. In fact, any closed hyperbolic $2k$-manifold $M$ can be realized as $\mathbb{H}^n/\Gamma$, where $\gamma \cong \pi_1(M)$ is a torsion-free discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, acting by deck transformations on $\mathbb{H}^n$. This proves the Dodziuk-Singer conjecture for closed hyperbolic manifolds. □

By Atiyah's formula 1.4.8 and Theorem 3.2.1, we have $(-1)^k\chi(M) = b_k^{(2)}(\mathbb{H}^n) > 0$. This proves the Hopf conjecture for closed hyperbolic $2k$-manifolds:

**Theorem 3.2.5.** *Let $M = \mathbb{H}^n/\Gamma$ be a closed hyperbolic $2k$-manifold. Then*

$$(-1)^k\chi(M) = b_k^{(2)}(\mathbb{H}^n) > 0.$$

Recall that the $k^{th}$ $\ell^2$-Betti number of $\mathbb{H}^n$, when viewed as the universal cover of $M = \mathbb{H}^n/\Gamma$, equals the von Neumann dimension of $\mathcal{H}^k_{(2)}(\mathbb{H}^n)$ with respect to $\Gamma \cong \pi_1(M)$, so the Euler characteristic depends on (and only on) the fundamental group of $M$ as expected.

## 3.3 Additional results

Let us mention some additional known cases of the Dodziuk-Singer conjecture. First, we have the following two theorems for closed manifolds with pinched sectional curvature, due to Jost and Xin.

**Theorem 3.3.1** ([22, Theorem 2.3])**.** *Let $\widetilde{M}$ be the universal cover of a closed Riemannian manifold $M$ with $\dim M \geq 4$ whose sectional curvature satisfies $-a^2 \leq K \leq -b^2$ for some $a, b > 0$. If $k \neq \frac{1}{2}\dim M$ and $(2k-1)a \leq (\dim M - 2)b$, then*

$$\mathcal{H}_k^{(2)}(\widetilde{M}) = 0.$$

The *Ricci curvature tensor* $\mathrm{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}$ of $(M, g)$ is the trace of the Riemann curvature tensor $R$; for each point $p \in M$, it is given by $\mathrm{Ric}_p(Y, Z) := \mathrm{tr}(X \mapsto R_p(X, Y)Z)$.

We say that the Ricci curvature of $(M, g)$ is *bounded from above by* $a > 0$ if $\mathrm{Ric}_p(v, v) \leq a$ for all unit tangent vectors $v \in T_pM$. In terms of sectional curvature, $\mathrm{Ric}_p(v, v)$ equals $(n-1)$ times the average of the sectional curvatures taken over the 2-dimensional subspaces containing $v$ for a unit tangent vector $v \in T_pM$ on an $n$-manifold $M$.

**Theorem 3.3.2** ([22, Theorem 2.1])**.** *Let $\widetilde{M}$ be the universal cover of a closed Riemannian manifold $M$ with $\dim M \geq 3$ whose sectional curvature satisfies $-a^2 \leq K \leq 0$ for some $a > 0$ and whose Ricci curvature satisfies Ric$(M) \leq -b^2$ for some $b > 0$. If $k \neq \frac{1}{2} \dim M$ and $k \leq \frac{b}{2a}$, then*

$$\mathcal{H}_k^{(2)}(\widetilde{M}) = 0.$$

Let $M$ be a complete Riemannian manifold. A *normal neighbourhood* of a point $p \in M$ is an open neighbourhood $U$ of $p$ such that there exists a open neighbourhood $V$ of the origin in the tangent space $T_pM$ such that for any $v \in V$, $tv \in V$ for all $t \in [-1, 1]$, and the exponential map restricted to $V$ induces a diffeomorphism $\exp_p : V \xrightarrow{\sim} U$. For any normal neighbourhood $V$ of a point $p \in M$, the *geodesic symmetry* is the diffeomorphism $\mathrm{sym}_p : U \to U$, $\mathrm{sym}_p : \exp_p(v) \mapsto \exp_p(-v)$ for any $v \in V$. A complete Riemannian manifold $M$ is a *locally symmetric space* if for any $p \in M$, there exists a normal neighbourhood $U \ni p$ such that the geodesic symmetry $\mathrm{sym}_p$ of $U$ is an isometry. The $\ell^2$-cohomology of such spaces has been shown to vanish outside the middle degree.

**Theorem 3.3.3** ([25, Corollary 5.16])**.** *Let $M$ be a closed locally symmetric Riemannian $n$-manifold with strictly negative sectional curvature with universal cover $\widetilde{M}$. Then*

$$\mathcal{H}_k^{(2)}(\widetilde{M}) = 0$$

*for all $k \neq \frac{n}{2}$.*

# Chapter 2

# Combinatorial analogues

Chapter 2 of this thesis is devoted to studying the Hopf and Dodziuk-Singer conjectures for nonpositively curved cell complexes, focusing on the combinatorial aspects of the problem. In Section 4, we cover the necessary background in combinatorics and metric geometry, including how the metric analogue of nonpositive curvature is a purely combinatorial condition for Euclidean cubical complexes. Using this, we present the combinatorial analogue of the Hopf conjecture, known as the Charney-Davis conjecture. In Section 5, we construct examples of nonpositively curved cubical complexes, known as the Danzer complex, and prove decomposition theorems on the homology groups of these spaces. In Section 6, we introduce the cellular version of $\ell^2$-cohomology and use this to study the Dodziuk-Singer conjecture for the universal cover of the Danzer complex, which are spaces modeled on right-angled Coxeter groups.

## 4 Preliminaries in combinatorics and metric geometry

In this section we first cover the necessary material from combinatorics and metric geometry, concluding with showing how nonpositive curvature for Euclidean cubical complexes is equivalent to a purely combinatorial condition on the links of vertices. Using a combinatorial analogue of the Gauss-Bonnet formula, we arrive at a combinatorial version of the Hopf conjecture for piecewise Euclidean manifolds, known as the Charney-Davis conjecture on flag triangulations of spheres. Finally, we tie this conjecture to some quantities familiar to combinatorialists.

## 4.1 Combinatorics of complexes

Let $\Delta$ be a simplicial complex. We denote $\mathcal{S}(\Delta)$ for the *face poset* of $\Delta$, consisting of the smiplices of $\Delta$, ordered by inclusion, and $\mathcal{S}_k(\Delta)$ for the $k$-simplices in $\Delta$. For example, $\mathcal{S}_0(\Delta) = V(\Delta)$. For a poset $\mathcal{P}$, the *order complex* $O(\mathcal{P})$ of $\mathcal{P}$ is the simplicial complex on the elements of $\mathcal{P}$, with the faces of $O(\mathcal{P})$ corresponding to the chains in $\mathcal{P}$. The *barycentric subdivision* of a simplicial complex is the order complex of its face poset. For any complex $\Delta$ we denote the $k$-skeleton by $\Delta^{(k)}$. All the complexes considered in this thesis are locally finite, i.e., every cell is disjoint from all but finitely many cells.

**Definition 4.1.1.** A simplicial complex $\Delta$ is a *flag complex* if every set of pairwise connected vertices spans a simplex in $\Delta$.

This is sometimes referred to as the "no $\Delta$-condition", as it is equivalent to having no hollow 2-simplex as an induced subcomplex of $\Delta$. Flag complexes are in bijection with simple graphs; given a flag complex $\Delta$, its 1-skeleton is a simple graph, and conversely one can form a flag complex from a simple graph $G$ by taking its *clique complex*, which is the simplicial complex on the vertices of $G$ with simplices corresponding to the cliques (induced complete subgraphs) of $G$. Clearly, any induced subcomplex of a flag complex is also flag. For example, all order complexes of posets, and hence barycentric subdivisions of simplicial complexes, are flag complexes. This shows that flagness does not restrict the topological type of a simplicial complex, and is a purely combinatorial condition.

For a simplicial complex $\Delta$ and $\sigma \in \mathcal{S}(\Delta)$, the *link* $\mathrm{lk}_\sigma(\Delta)$ of $\sigma$ in $\Delta$ is defined as $\mathrm{lk}_\sigma(\Delta) := \{\tau \in \mathcal{S}(\Delta) \mid \sigma \cap \tau = \varnothing,\ \sigma \cup \tau \in \mathcal{S}(\Delta)\}$, and the *star* $\mathrm{st}_\sigma(\Delta)$ of $\sigma$ in $\Delta$ as $\mathrm{st}_\sigma(\Delta) := \{\tau \in \mathcal{S}(\Delta) \mid \sigma \cup \tau \in \mathcal{S}(\Delta)\}$. The *open star* is defined as the pair $(\mathrm{st}_\sigma(\Delta), \mathrm{lk}_\sigma(\Delta))$. For two simplicial complexes $K, L$ on disjoint ground sets, their *join* $K * L$ is the simplicial complex with the ground set being the disjoint union of those of $K$ and $L$, and the faces of $K * L$ are all sets of the form $\sigma \cup \tau$, where $\sigma \in K$ and $\tau \in L$. The *cone* $C\Delta$ of a simplicial complex $\Delta$ is $v * \Delta$ for a vertex $v$ disjoint from the ground set of $\Delta$, and the *suspension* $\Sigma\Delta$ of $\Delta$ is defined as $\mathbb{S}^0 * \Delta$. Topologically, these coincide with the usual notions of cones and suspensions of topological spaces.

A locally compact topological space $X$ is a *homology $n$-manifold* with coefficients in $R$ if it satisfies the local properties of a manifold in terms of homology, i.e., if $H_i(X, X - x, R) = 0$ for $i \neq n$ and $H_n(X, X - x, R) = R$ for all points $x \in X$. In this thesis, we take the definition to be with respect to $R = \mathbb{Q}$, i.e., with rational coefficients. For a simplicial complex $\Delta$, this local condition is equivalent to the link $\mathrm{lk}_\sigma(X)$ having the homology of $\mathbb{S}^{n-k-1}$ for each $k$-simplex $\sigma \in X$. A simplicial complex $X$ is a *generalized homology $n$-sphere*, which we abbreviate to $\mathsf{GHS}^n$, if it is a homology $n$-manifold that has the homology of $\mathbb{S}^n$, again with rational coefficients. If $S$ is a $\mathsf{GHS}^n$, then $\mathrm{lk}_v S$ is a $\mathsf{GHS}^{n-1}$ for every vertex $v$ in $S$. A pair $(D, \partial D)$ is a *generalized homology $n$-disk*, abbreviated $\mathsf{GHD}^n$, if it is a homology $n$-manifold with boundary and $H_i(D, \partial D) = 0$ if $i \neq n$ and $H_n(D, \partial D) = \mathbb{Q}$ . For an $n$-dimensional simplicial complex to be a homology $n$-manifold, it suffices that the vertex links are generalized homology $(n-1)$-spheres.

## 4.2 Metric geometry & CAT(0) spaces

In this section we cover the necessary background in metric geometry needed for our purposes, with the aim of understanding how to construct examples of aspherical spaces through cubical complexes.

Let $(X, d)$ be a metric space and $I \subset \mathbb{R}$ a non-empty interval. The length $\ell(\gamma) \in [0, \infty]$ of a curve $\gamma : I \to X$ is given by $\ell(\gamma) := \sup \sum_{i=1}^{k-1} d(\gamma(t_i), \gamma(t_{i+1}))$, where the suprenum is taken over the set of partitions of $I$. The space $(X, d)$ is a *length space* if $d(x, y) = \inf \ell(\gamma)$, where the infimum is over all curves $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\ell(\gamma) < \infty$. A map $f : (X, d) \to (X', d')$ of metric spaces is an *isometry* onto its image if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

For points $x, y \in X$, a *geodesic* $\gamma$ from $x$ to $y$ is a continuous map $\gamma : [0, d(x, y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in [0, d(x, y)]$. The space $(X, d)$ is a *(uniquely) geodesic* space if any two points are connected by a (unique) geodesic. Clearly, every geodesic space is a length space.

To define notions of bounded curvature, we first define the *model space $M^n_\kappa$ of curvature $\kappa$* as the unique simply connected and complete Riemannian manifold with constant sectional curvature $\kappa$. If $\kappa < 0$, then $M^n_\kappa$ is obtained from hyperbolic space $\mathbb{H}^n$ by scaling the distance function by $\frac{1}{\sqrt{-\kappa}}$, and if $\kappa > 0$, then $M^n_\kappa$ is the standard $n$-sphere with the distance function scaled by $\frac{1}{\sqrt{\kappa}}$.

Let $(X, d)$ be a geodesic space. For points $x, y, z \in X$, a *triangle* $\Delta(x, y, z)$ formed by these points in $X$ consists of some geodesic segments, called sides, connecting the points pairwise. For such a triangle $\Delta(x, y, z)$, a *comparison triangle* $\bar{\Delta}(\bar{x}, \bar{y}, \bar{x}) \subset M^2_\kappa$ is a triangle with vertices $\bar{x}, \bar{y}, \bar{x}$ such that the distances of the sides of $\Delta$ equal the lengths of the corresponding segments in $M^2_\kappa$ (i.e., $d(x,y) = \bar{d}(\bar{x},\bar{y})$, where $\bar{d}$ is the metric on $M^2_\kappa$, and similarly for $d(x,z)$, $d(y,z)$). For a comparison triangle for some points $x,y,z \in X$ to exist, we require that the perimeter satisfies $d(x,y)+d(y,z)+d(z,x) < \frac{2\pi}{\sqrt{\kappa}} =: D_\kappa$ when $\kappa > 0$. The quantity $D_\kappa$ is the *diameter* of $M^2_\kappa$, and is defined to be infinite for $\kappa \leq 0$, so in particular comparison triangles always exists for $\kappa \leq 0$. If a comparison triangle in $M^2_\kappa$ exists, then it is unique up to isometry. For proofs of these facts we refer to [5, Chapter I.2]. For any point $p \in \Delta$ on a side $xy$, the *comparison point* $\bar{p}$ in $\bar{\Delta}$ is the point on the corresponding side $\bar{x}\bar{y}$ of $\bar{\Delta}$ such that $d(x,p) = \bar{d}(\bar{x},\bar{p})$ and $d(y,p) = \bar{d}(\bar{y},\bar{p})$.

**Definition 4.2.1.** Let $(X,d)$ be a metric space and $\Delta \subset X$ any geodesic triangle in $X$. Let $\bar{\Delta} \subset M^2_\kappa$ be a comparison triangle for $\Delta$. For $\kappa \leq 0$, $X$ is a $\mathrm{CAT}(\kappa)$ *space* if for any $x,y \in \Delta$, we have $d(x,y) \leq \bar{d}(\bar{x},\bar{y})$ for the comparison points $\bar{x},\bar{y} \in \bar{\Delta} \subset M^2_\kappa$. For $\kappa > 0$, $X$ is $\mathrm{CAT}(\kappa)$ if all points less than $D_\kappa$ apart are joined by a geodesic, and the above condition on comparison points holds for all geodesic triangles of perimeter $< 2D_\kappa$.

In the $\kappa = 0$ case, geometrically this signifies that triangles in $(X,d)$ are at least as slim as Euclidean space, which agrees with our intuition from hyperbolic space; see Figure 2.1 for a visualization.

The following basic fact characterizes the relationship between $\mathrm{CAT}(\kappa)$ spaces for varying $\kappa$.

**Theorem 4.2.2** ([5, p. 165])**.**

*(i) If $X$ is a $\mathrm{CAT}(\kappa)$ space, then $X$ is $\mathrm{CAT}(\kappa')$ for any $\kappa' \geq \kappa$.*

*(ii) If $X$ is a $\mathrm{CAT}(\kappa')$ space for every $\kappa' > \kappa$, then it is a $\mathrm{CAT}(\kappa)$ space.*

For example, hyperbolic space $\mathbb{H}^n$ is a $\mathrm{CAT}(0)$ space by part (i) of Theorem 4.2.2.

**Definition 4.2.3.** A metric space $(X,d)$ is *locally* $\mathrm{CAT}(\kappa)$ or *has curvature* $\leq \kappa$ if every $x \in X$ has an open neighbourhood $U \ni x$ that is a $\mathrm{CAT}(\kappa)$ space with the induced metric.

If $(X,d)$ is locally $\mathrm{CAT}(0)$, we say that $(X,d)$ has *nonpositive curvature.*

Recall that the sectional curvature of a Riemannian manifold $M$ is bounded by $\kappa$ if at every point $p \in M$ and every 2-dimensional subspace $\prod \subseteq T_pM$ the sectional curvature $K$ satisfies $K(\prod) \leq \kappa$. In the smooth case, the following classical result of E. Cartan relates the notion of $\mathrm{CAT}(\kappa)$ curvature to the sectional curvature of a Riemannian manifold.

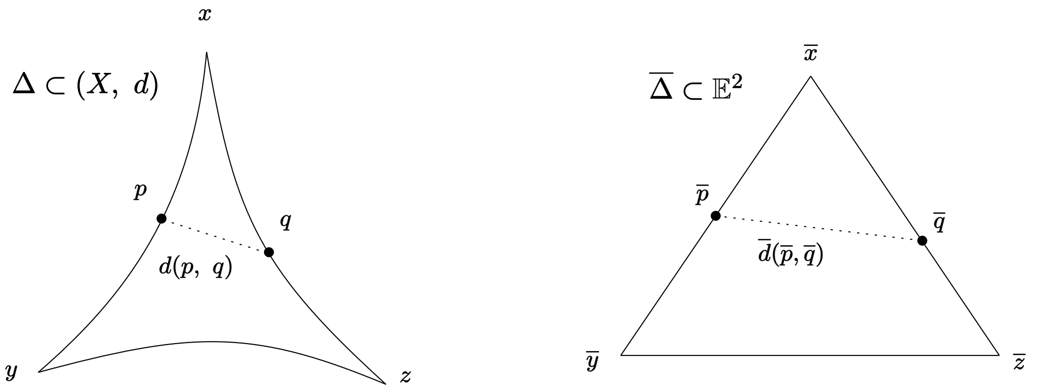


**Figure 2.1.** Example of a geodesic triangle $\Delta$ in a CAT(0)-space $(X, d)$ and a comparison triangle $\bar{\Delta}$ in the Euclidean plane with the Euclidean metric $\bar{d}$: for any points $p, q \in \Delta$, we have $d(p, q) \leq \bar{d}(\bar{p}, \bar{q})$.

**Theorem 4.2.4** ([5, p. 173])**.** *A Riemannian manifold $(M, g)$ has curvature $\leq \kappa$ in the* CAT$(\kappa)$ *sense if and only if the sectional curvature of $M$ is bounded above by $\kappa$.*

Now, let us turn our attention to CAT(0) spaces, which is what we will mostly be focusing on throughout the rest of the thesis.

**Proposition 4.2.5.** *Every* CAT(0) *space $(X, d)$ is uniquely geodesic.*

*Proof.* Fix distinct points $x, y \in X$ and let $\gamma, \tau$ be geodesics connecting $x$ and $y$. Let $p, q$ be points on $\gamma, \tau$ respectively such that $d(x, p) = d(x, q)$, so that the comparison triangle $\bar{\Delta}(\bar{x}, \bar{y}, \bar{p})$ of the triangle $\Delta(x, y, p)$ is degenerate, with the geodesic side $xy$ equal to $\gamma$ and $\bar{p} = \bar{q}$, as $\mathbb{R}^2$ is uniquely geodesic. By the CAT(0) inequality we get

$$0 \leq d(p, q) \leq d(\bar{p}, \bar{q}) = 0,$$

and hence $p = q$. As the choice of the points $p$, $q$ on $\gamma, \tau$ was arbitrary, $\gamma = \tau$. $\square$

A metric space $(X, d)$ is *proper* if all closed metric balls $B(x_0, r) = \{x \in X \mid d(x_0, x) \leq r\}$ are compact. A length space is proper if and only if it is complete and locally compact, meaning that every point has a compact neighbourhood [5, Corollary 3.8]. All complete CAT(0) spaces are proper.

**Proposition 4.2.6.** *Any uniquely geodesic proper metric space is contractible. In particular, all* CAT(0) *spaces are contractible.*

*Proof.* If $(X, d)$ is a uniquely geodesic space, for any fixed point $x_0 \in X$ we may define the map $g_{x_0} : X \times [0, 1] \to X$ which sends $(x, t)$ to the point on the unique geodesic $\gamma(x, x_0)$ which is at a distance of $td(x_0, x)$ from $x$ to $x_0$. Applying the Arzelá-Ascoli lemma on equicontinuous sequences of maps of compact metric spaces (which is where we need the properness assumption), it follows that these unique geodesics vary continuously with respect to their endpoints [5, Corollary 3.13], making the map continuous. Hence $g_{x_0} : X \times [0,1] \to X$ gives a continuous retraction of $X$ to $x_0$, showing that $X$ is contractible. $\square$

A metric $d$ on a geodesic space $X$ is *convex* if all geodesics $\gamma : [0,a] \to X, \tau : [0,b] \to X$ with $\gamma(0) = \tau(0)$ satisfy $d(\gamma(ta), \tau(tb)) \leq td(\gamma(a), \gamma(b))$ for all $t \in [0,1]$. The metric space $(X,d)$ is *locally convex* if for every $x \in X$ there exists an open neighbourhood $U \ni x$ where the induced metric is convex. In this case, $X$ is locally contractible, and hence there exists a universal cover $p : \widetilde{X} \to X$. The projection $p$ is a local homeomorphism, and given a path $c : [0,1] \to \widetilde{X}$, we may define the length $\ell(c)$ of $c$ to be the length of the projected curve $p \circ c$ in $X$. If $\widetilde{X}$ is Hausdorff, defining $\tilde{d}(\tilde{x}, \tilde{y})$ for points $\tilde{x}, \tilde{y} \in \widetilde{X}$ to be the inf $\ell(c)$ for paths $c$ in $\widetilde{X}$ joining $\tilde{x}$ and $\tilde{y}$ gives a metric on $\widetilde{X}$, and with this metric $p$ becomes a local isometry [5, Proposition 3.25].

In this setup, Gromov [17] has proved a variation of the Cartan-Hadamard theorem for metric spaces.

**Theorem 4.2.7** (Metric Cartan-Hadamard Theorem). *Let $(X,d)$ be a complete connected metric space. If d is locally convex, then $(\widetilde{X}, \tilde{d})$ is a uniquely geodesic space. Furthermore, if $(X,d)$ is locally $\mathrm{CAT}(\kappa)$, then $(\widetilde{X}, \tilde{d})$ is a $\mathrm{CAT}(\kappa)$ space.*

In fact, the difference between CAT(0) spaces and spaces of nonpositive curvature is purely a topological one:

**Theorem 4.2.8.** *A length space is* CAT(0) *is and only if it is nonpositively curved and simply connected.*

Proofs of the above results can be found in [5, p. 193]. As CAT(0) spaces are contractible, we have the following important corollary.

**Corollary 4.2.9.** *A complete geodesic space with nonpositive curvature is aspherical.*

Next, we define the notion of convex polytopes in $M^n_\kappa$ for $\kappa \in \{-1,0,1\}$. When $\kappa = 0$, a convex polytope is a compact intersection of finitely many half-spaces, which we also refer to as Euclidean cells. When $\kappa = 1$ and $M^n_\kappa = \mathbb{S}^n$, a convex polytope is the intersection of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with a convex polyhedral cone in $\mathbb{R}^{n+1}$, i.e., a set of the form $\{\sum_{i=0}^n \lambda_i v_i \mid \lambda_i \geq 0,\ v_i \in \mathbb{R}^{n+1}\}$, with the additional condition that this cone contains no line in $\mathbb{R}^{n+1}$. In the hyperboloid model of hyperbolic space $M^n_{-1} = \mathbb{H}^n \subset \mathbb{R}^{n,1}$, a convex polytope is the intersection of a convex polydedral cone $C \subset \mathbb{R}^{n,1}$ such that $C \setminus \{0\}$ is contained in the interior of the set $\{x \in \mathbb{R}^{n,1} \mid \langle x, x\rangle \geq 0,\ x_{n+1} \geq 0\}$. Recall that $\mathbb{R}^{n,1}$ is $\mathbb{R}^{n+1}$ equipped with the symmetric bilinear form $\langle x, y\rangle = x_1y_1 + \ldots + x_ny_n - x_{n+1}y_{n+1}$, and the hyperboloid model of $\mathbb{H}^n$ is the component of the hypersurface $\langle x, x\rangle = -1$ contained in $x_{n+1} > 0$. All these polytopes inherit a metric from the ambient space.

Let us now consider a more combinatorially flavored way to construct metric spaces of nonpositive curvature, namely $M_\kappa$ polyhedral complexes, or $M_\kappa$ complexes for short. An *$M_\kappa$ polyhedral complex* is a collection of convex polyhedra in $M_\kappa$, with faces of distinct polyhedra possibly identified by isometries. In other words, we glue convex polyhedra along isometries of their faces. Such a complex is naturally endowed with a piecewise metric coming from the the metric structure of the cells: given a $M_\kappa$ complex $X$ and a path $\gamma : [0, 1] \to X$, we define the length $\ell_\kappa(\gamma)$ of $\gamma$ as

$$\ell_\kappa(\gamma) = \sup\big\{\sum_{i=0}^{k} d_\kappa(\gamma(t_{i-1}), \gamma(t_i))\big\},$$

where $(t_0, \ldots, t_k)$ runs over all subdivision of the interval $[0, 1]$ such that $\gamma(t_{i-1})$ and $\gamma(t_i)$ are contained in the same cell of $X$, and $d_\kappa$ denotes the metric of the cell in question. Connected $M_\kappa$ complexes are length space: for all $x, y \in X$, $d(x, y)$ is the infimum of $\ell_\kappa(\gamma)$ over all paths $\gamma$ from $x$ to $y$. Clearly, the distance function $d$ on a $M_\kappa$ complex $X$ restricts to the distance function coming from $M^n_\kappa$ for any of the cells in $X$. A *Euclidean complex* is a $M_0$ polyhedral complex, so a complex of convex Euclidean polyhedra, or Euclidean cells. Similarly, a *piecewise spherical* cell complex is a $M_1$ polyhedral complex.

An important result on $M_\kappa$ complexes is that such a complex is a complete geodesic metric space if the complex has only finitely many isometry types of cells [5, Theorem 7.50]. In particular, this is case when $X$ is a finite complex or a covering space of a finite complex with the induced metric and lifted cellular structure. From now on, all $M_\kappa$ complexes will be of this type. For us, the main class of $M_\kappa$ complexes are the following.

**Definition 4.2.10.** A *Euclidean cubical complex* is a cell complex obtained by gluing standard Euclidean cubes $[-1, 1]^k$ of possibly varying dimension $k \in \mathbb{N}$ along isometries of their faces.

From now on, we will work only with Euclidean cubical complexes, and refer to these spaces simply as cubical complexes. For a convex $M_\kappa$-polytope $X$ and a vertex $v \in X$, we define the *link* $\mathrm{lk}_v X$ of $v$ in $X$ to be the set of unit vectors at $v$ that point into $X$. A link has a natural structure of a spherical cell complex. For a $M_\kappa$ polyhedral complex $X$, we define the link of $v$ in $X$ to be the union of links of $v$ for all cells containing $v$. Intuitively, the link is topologically the intersection of $X$ and a small sphere centered at $v$. If $X$ is an $n$-manifold, then $\mathrm{lk}_v X$ is homeomorphic to $\mathbb{S}^{n-1}$ for all vertices $v$ in $X$. For cubical complexes, links are topologically simplicial complexes.

The following lemma of Gromov is a critical component of studying the Hopf conjecture from a combinatorial viewpoint, as it translates nonpositive curvature of a Euclidean cubical complexes to a purely combinatorial statement on the structure of vertex links in the complex. A proof can be found in [11, Appendix I.6.].

**Lemma 4.2.11** (Gromov's Lemma)**.** *A Euclidean cubical complex $X$ with the natural piecewise Euclidean metric is non-positively curved if and only if the link of each vertex in $X$ is a flag complex.*

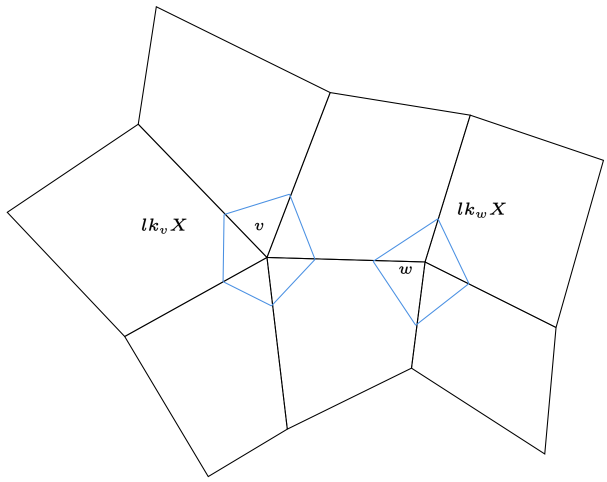


**Figure 2.2.** Local structure of a CAT(0) Euclidean cubical complex; the vertex links are flag complexes.

Let $C$ be a convex polyhedral cone in $\mathbb{R}^n$. To such a cone we may associate a spherical $(n-1)$-cell $\sigma$ by intersecting the cone with the unit sphere, i.e., $\sigma = \mathbb{S}^{n-1} \cap C$. Any such cone has a dual cone $C^\vee$, obtained by taking all nonnegative linear combinations of the outward pointing normal vectors of the supporting hyperplanes of $C$. Define $\sigma^\vee := \mathbb{S}^{n-1} \cap C^\vee$ to be the dual cell of $\sigma$. The *angle of* $C$ at the origin, denoted $\alpha(\sigma)$, is defined as

$$\alpha(\sigma) = \frac{\mathrm{vol}(\sigma)}{\mathrm{vol}(\mathbb{S}^{n-1})},$$

the normalized volume of $\sigma$. The *exterior angle* $\alpha^\vee$ of $C$ is defined as $\alpha^\vee(\sigma) = \alpha(\sigma^\vee)$. For a Euclidean $n$-cell $X$, we may choose any interior point $p \in \mathrm{int}(X)$ to be the origin, and define the dual cell $X^\vee$ to be the convex cell whose vertices are the normal vectors of codimension one faces of $P$. We may radially project the boundary of $P^\vee$ onto $\mathbb{S}^{n-1}$ to cellulate $\mathbb{S}^{n-1}$. Note that for a Euclidean cell $X$, we have $\sum_{v\in V(X)} \alpha^\vee(\mathrm{lk}_v X) = 1$, as $\alpha$ is normalized to give $\alpha(\mathbb{S}^{n-1}) = 1$.

**Definition 4.2.12.** A spherical $(n-1)$-cell $\sigma$ in $\mathbb{S}^{n-1}$ is *all right* if the vertices of $\sigma$ form an orthonormal basis for $\mathbb{R}^n$. As spherical cell complex is all right if each of its cells is an all right cell. In particular, $\mathrm{lk}_v X$ is an all right spherical complex if $X$ is a cubical complex.

For an all right cell we of course have $\sigma^\vee = \sigma$. Now, suppose $X$ is a cubical complex. Then we have

$$\begin{aligned}\chi(X) &= \sum_{\sigma\in\mathcal{S}(X)} (-1)^{\dim\sigma} \\ &= \sum_{\sigma\in\mathcal{S}(X)} (-1)^{\dim\sigma} \sum_{v\in V(X)} \alpha^\vee(\mathrm{lk}_v X) \\ &= \sum_{v\in V(X)} (1 + \sum_{\sigma\in\mathcal{S}(\mathrm{lk}_v X)} (-1)^{\dim\sigma+1}\alpha^\vee(\sigma)),\end{aligned}$$

where we have reversed the order of summation. For a finite piecewise spherical cell complex $L$, define

$$\kappa(L) = 1 + \sum_{\sigma\in\mathcal{S}(L)} (-1)^{\dim\sigma+1}\alpha^\vee(\sigma).$$

When $\sigma$ is an all-right spherical (as in the case of links of cube complexes), $\mathbb{S}^{n-1}$ is cellulated by $2^n$ copies of $\sigma$. As $\sum\alpha(\sigma) = 1$ ($\alpha$ is normalized so that $\alpha(\mathbb{S}^{n-1}) = 1$), we have $\alpha^\vee(\sigma) = \alpha(\sigma) = (\frac{1}{2})^n$, and hence

$$\kappa(L) = \sum_{i=-1}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i(L)$$

for an all right spherical complex, where $f_{-1}(L) = 1$. Applying this to the expression for $\chi(X)$ derived above, we have the following result.

**Theorem 4.2.13** (Combinatorial Gauss-Bonnet theorem)**.** *Let $X$ be a Euclidean cubical complex. Then*

$$\chi(X) = \sum_{v\in V(X)} \kappa(lk_v X),$$

*where* $\kappa(\Delta) = \sum_{i=-1} \left(-\frac{1}{2}\right)^{i+1} f_i(\Delta)$ *with* $f_{-1} = 1$.

### 4.3 The Charney-Davis and Gal conjectures

This section covers the combinatorial analogue of the Hopf conjecture introduced by Charney and Davis in [7]. First, let us gather some of the material developed in the previous section from the viewpoint of the Hopf conjecture.

In the combinatorial analogue of the Gauss-Bonnet formula, Theorem 4.2.13, the summand associated to the vertex links can be thought of as an analogue of the Chern-Gauss-Bonnet integrand. We define this value as the *Charney-Davis quantity* $\kappa(L)$ of a simplicial complex $L$:

$$\kappa(L) = \sum_{i=-1}^{\dim L} (-\frac{1}{2})^{i+1} f_i(L).$$

As before, we set $f_{-1}(L) = 1$. Applying Gromov's condition 4.2.11 for nonpositive curvature to a Euclidean cubical manifold and using the combinatorial Gauss-Bonnet formula, we see that if $(-1)^k\kappa(L) \geq 0$ for any flag $(2k-1)$-sphere $L$, the Hopf conjecture for such manifolds follows. Hence, we arrive at the following conjecture:

**Conjecture 4.3.1** (Charney-Davis conjecture)**.** *Let $L$ be a $(2k-1)$-dimensional flag sphere. Then*

$$(-1)^k\kappa(L) \geq 0.$$

This is in contrast to the usual Chern-Gauss-Bonnet Theorem for smooth manifolds, where, as shown by Geroch in [16], the Hopf conjecture cannot be deduced in dimensions six and up by simply considering the sign of the integrand for the Euler characteristic. In fact, in the context of Euclidean cubical complexes, the two conjectures are equivalent, thanks to a construction known as the Danzer complex $\mathfrak{D}(L)$ associated to a simplicial complex $L$, which is the main focus of Section 5. This is a nonpositively curved Euclidean cube complex which is a subcomplex of the $n$-cube $C_n \subset \mathbb{R}^n$, where $n$ is the number of vertices in $L$, with the property that all the vertex links in $\mathfrak{D}(L)$ are isomorphic to $L$. Hence, by Theorem 4.2.13 we have

$$\chi(\mathfrak{D}(L)) = \sum_{v\in V(\mathfrak{D}(L))} \kappa(\mathrm{lk}_v(\mathfrak{D}(L))) = 2^n\kappa(L).$$

Hence, the Euler characteristic of $\mathfrak{D}(L)$ and $\kappa(L)$ have the same sign. If the Hopf conjecture $(-1)^k\chi(\mathfrak{D}(L))$ holds for Euclidean cubical $2k$-manifolds, it must hold for $\mathfrak{D}(L)$, and thus each of the links must satisfy the Charney-Davis inequality.

Thus, we see how the Hopf conjecture for cubical manifolds becomes a purely combinatorial statement about the face numbers of flag spheres. Let us now tie this conjecture in to some of the usual quantities combinatorialists are interested in when studying simplicial complexes. Recall that the $f$-polynomial of $\Delta$ is defined as $f_\Delta(t) = \sum_{i=0} f_{i-1} t^i$, where $f_i$ are the face numbers of $\Delta$. The $h$-vector $(h_0, \dots, h_d)$ of $\Delta$ is defined by the equation

$$\sum_{i=0}^{d+1} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i},$$

and the $h$-polynomial of $\Delta$ is defined as $h_\Delta(t) = \sum_{i=0}^{d} h_i t^i$.

In many cases, the $h$-vector encodes the combinatorial information of $\Delta$ in a more robust way than the $f$-vector; for example, the famous Dehn-Sommerville relations for a simplicial $(d-1)$-sphere $\Delta$ (or more generally for homology spheres) can be expressed as $h_i(\Delta) = h_{d-i}(\Delta)$ for $i = 0, \dots, d$. Note that $h_L(-1) = 2^d f_L(-\frac{1}{2}) = 2^d \kappa(L)$, so in particular these quantities have the same sign, and hence the Charney-Davis conjecture is equivalent to $(-1)^d h_L(-1) \geq 0$.

An even stronger version of this conjecture, introduced by Gal in [15], is on the non-negativity of the $\gamma$-vector of $\Delta$. If $L$ is a (homology) sphere, then the $h$-vector of $\Delta$ is symmetric about the middle by the Dehn-Sommerville equations. Hence, the $h$-polynomial of $\Delta$ can be written in a basis of symmetric polynomials

$$h_L(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i},$$

where the coefficients $\gamma_i$ define the $\gamma$-vector of $L$ as well as its $\gamma$-polynomial $\gamma^L(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i^L t^i$. Gal's conjecture concerns these coefficients for flag homology spheres.

**Conjecture 4.3.2** (Gal's conjecture)**.** *Let $L$ be a flag homology sphere. Then $\gamma_i^L \geq 0$ for all $i = 0, \dots, \lfloor \frac{d}{2} \rfloor$.*

As the Charney-Davis quantity $(-1)^d h_L(-1)$ is exactly the coefficient of highest degree in $\gamma^L(t)$, it follows that this is a strengthening of the Charney-Davis conjecture.

The sign of the Charney-Davis quantity is known some special cases, including when $L$ is the boundary complex of a simple $2k$-polytope satisfying an additional local convexity property that is stronger than flagness. In this case Reiner and Leung [24] proved that the Charney-Davis quantity equals the signature of a toric variety associated to $L$, and using the Hirzebruch signature theorem and the additional convexity property, showed that the $(-1)^k\kappa(L) \geq 0$. Additionally, Babson has observed that based on earlier work of Stanley, the conjecture also holds for boundary complexes of simplicial $2k$-polytopes which are order complexes of posets, including barycentric subdivisions of polytopes.

# 5 The Danzer complex

The aim of this section is to construct the Danzer complex $\mathfrak{D}(L)$ and the Davis complex $\Sigma_L$ associated to a flag simplicial complex $L$. The complex $\mathfrak{D}(L)$ is a nonpositively curved Euclidean cube complex with $\Sigma_L$ as its universal cover. To construct $\Sigma_L$, we first cover some background in Coxeter groups and their topology. The standard reference on this topic is [11]. We then prove decomposition results for the homology of the Danzer complex $\mathfrak{D}(L)$, and use this to reformulate the Charney-Davis conjecture in terms of the Betti numbers of the Stanley-Reisner ring of $L$.

## 5.1 Right-angled Coxeter groups and the Davis complex

The theory of Coxeter groups and their connection to topology and geometry is particularly rich, and there are many interesting topological constructions that work in great generality. However, in this thesis we are only interested in the case of right-angled Coxeter groups (RACG), which are in bijection with finite simplicial graphs, i.e., a graph with no loops or multi-edges.

**Definition 5.1.1.** Let $G = (V, E)$ be a finite simplicial graph. The *right-angled Coxeter group* $W_G$ associated to $G$ is defined by the presentation

$$W_G = \langle V(G) \mid v^2 = e \ \forall v \in V,\ [v, w] = e \text{ if } (v, w) \in E\rangle.$$

Immediately from the definition we note that if the vertices $v$ and $w$ are not adjacent, then $vw$ has infinite order, and that $W_G$ is abelian if and only if $G$ is complete, which in turn is equivalent to $W_G$ being finite.

From now on, we speak interchangeably of flag complexes and graphs; if $L$ is a flag complex, then $W_L$ denotes the right-angled Coxeter group defined by its 1-skeleton. For any induced subcomplex $A \prec L$, $W_A$ is isomorphic to the subgroup of $W_L$ generated by $V(A)$. Such subgroups are called *special subgroups* of $W_L$. In the case where $A = \sigma$ is a simplex in $L$, in which case $W_\sigma$ is finite, the subgroups $W_\sigma \leq W_L$ are called *spherical subgroups*. A *spherical coset* in $W_L$ is defined as a coset of a spherical subgroup, i.e., of the form $wW_\sigma$ for some $w \in W_L$ and $\sigma \in \mathcal{S}(L)$. The set of all spherical cosets is denoted by $W_L\mathcal{S}(L) = \cup_{\sigma\in\mathcal{S}(L)} W_L/W_\sigma$, partially ordered by inclusion of cosets. The group $W_L$ acts on $W_L\mathcal{S}(L)$ by $w : w'W_\sigma \mapsto (ww')W_\sigma$, with the quotient poset equal to $\mathcal{S}(L)$. Now, we come to the definition of one of the main topological spaces we are interested in for the rest of this thesis.

**Definition 5.1.2.** The *Davis complex* $\Sigma_L$ of the right-angled Coxeter group $W_L$ associated to a finite flag complex $L$ is the order complex of the poset $W_L\mathcal{S}(L)$.

The action of $W_L$ on $W_L\mathcal{S}(L)$ induces a simplicial action of $W_L$ on $\Sigma_L$. As the stabilizer of each cell is a conjugate of a special subgroup, which are finite by definition, the action is proper. If we denote by $K_L$ the order complex of $\mathcal{S}(L)$, with the empty set as the unique minimal element, then $K_L$ is a cone over the barycentric subdivision on $L$, and the orbit space of $W_L$ on $\Sigma_L$ is $K_L$, i.e., $\Sigma_L/W_L = K_L$. The inclusion $\mathcal{S}(L) \hookrightarrow W_L\mathcal{S}(L)$ induces an inclusion of simplicial complexes $K_L \hookrightarrow \Sigma_L$, and viewed as a subset of $\Sigma_L$ under this inclusion, $K_L$ is called the *fundamental chamber* of $\Sigma_L$. Similarly for any induced subcomplex $A \prec L$ we have inclusions $W_A \hookrightarrow W_L$ and $\Sigma_A \hookrightarrow \Sigma_L$, whose image is referred to as a *special subcomplex* of $\Sigma_L$. Note that whenever we consider the complex $\Sigma_L$, it is implicit that $L$ is a flag complex, with $L$ being the clique complex of the graph which defines $W_L$.

If the right-angled Coxeter group $W_L$ decomposes as $W_L = W_{L_1} \times W_{L_2}$, with the elements generated by $V(L_1)$ commuting with those generated by $V(L_2)$, then a subset $A_1 \cup A_2$ of $V(L_1) \cup V(L_2)$ is spherical if and only if $A_i \subset V(L_i)$ is spherical for $i = 1, 2$. In this case $L$ decomposes as $L = L_1 * L_2$.

Let us consider some examples.

**Example 5.1.3.**

**Single vertex:** If $L$ is a single vertex $v$, then $W_L \cong \mathbb{Z}_2$ and $\Sigma_L$ can be identified with the interval $[-1, 1] \subset \mathbb{R}$, with $W_L$ acting by reflections across the origin.

**Products and joins:** If $L = L_1 * L_2$, then $W_L = W_{L_1} \times W_{L_2}$ and $\Sigma_L = \Sigma_{L_1} \times \Sigma_{L_2}$.

**Simplices:** If $L = \Delta^k$, a $k$-simplex, then as $\Delta_k = v_0 * \ldots * v_k$, we get $\Sigma_{\Delta^k} = \prod_{i=0}^{k} \Sigma_{v_i} \cong [-1, 1]^{k+1}$.

**Disjoint unions and points:** If $L = L_1 \sqcup L_2$, then $K_L = K_{L_1} \bigvee K_{L_2}$, where $\bigvee$ denotes the one-point union, with the base point corresponding to $\varnothing \in \mathcal{S}(L_1) \cap \mathcal{S}(L_2)$. If $L$ is the disjoint union of $k$ points, then $\Sigma_L$ is an infinite tree where each vertex has $k$ neighbours. In particular for $\mathbb{S}^0 = v_0 \bigvee v_1$ we have $\Delta_{\mathbb{S}^0} \cong \mathbb{R}$.

**Cones and suspensions:** By the previous example, if $L$ is a cone $L = v * K$, then $\Sigma_L = \Sigma_v \times \Sigma_K = [-1,1] \times \Sigma_K$. Similarly, as the suspension $\mathrm{susp}(L)$ of a complex is the same as $\mathbb{S}^0 * L$, we get $\Sigma_{\mathrm{susp}(L)} = \mathbb{R} \times \Sigma_L$.

For a $k$-simplex $\Delta^k = v_0 * \ldots * v_k$, we have $W_{\Delta^k} \cong (\mathbb{Z}_2)^{k+1}$ and $\Sigma_{\Delta^k} \cong [-1,1]^{k+1}$, with $W_{\Delta^k}$ acting simply transitively on the vertices of the cube $[-1,1]^{k+1}$. For an arbitrary flag complex $L$, for each $k$-simplex $\sigma \in \mathcal{S}(L)$ we have a special subcomplex $\Sigma_\sigma \cong [-1,1]^{k+1} \hookrightarrow \Sigma_L$, which induces a decomposition of $\Sigma_L$ into a collection of subcomplexes $\{w\Sigma_\sigma\}$ for $\sigma \in \mathcal{S}(L)$ and $w \in W_L$, with each subcomplex isomorphic to a cube. Hence, we get a regular cubulation of $\Sigma_L$, where the poset of cells of $\Sigma_L$ is isomorphic to $W_L\mathcal{S}(L)$, when ordered with respect to inclusion. For this particular cellulation of $\Sigma_L$ by Euclidean cubes, the link of each vertex in $\Sigma_L$ is $L$.

Identifying each of these cubes in $\Sigma_L$ with the corresponding standard Euclidean cube of edge length 2 gives a piecewise Euclidean metric on $\Sigma_L$, with the distance $d(a,b)$ of two points $a, b \in \Sigma_L$ defined as the infimum of the lengths of piecewise linear paths in $\Sigma_L$ from $a$ to $b$. With this choice of metric, $\Sigma_L$ becomes a geodesic space, i.e., for any two points $a, b \in \Sigma_L$ there exists a piecewise linear path of length $d(a,b)$ between them. Furthermore, the link of each vertex in this cubical structure is isomorphic to $L$. Now, one can show that $\Sigma_L$ is always a simply connected space [11, Lemma 7.3.5.]. Applying Gromov's lemma 4.2.11 and Theorem 4.2.8, we get that $\Sigma_L$ is a CAT(0) cube complex whenever $L$ is a flag complex, and hence a CAT(0) topological manifold if $L$ is a flag sphere.

## 5.2 The Danzer complex $\mathfrak{D}(L)$

In this subsection we describe a particularly nice way to construct a piecewise Euclidean cubical complex $\mathfrak{D}(L)$, known as the Danzer complex of $L$, which has the key property that the link of every vertex of $\mathfrak{D}(L)$ is isomorphic to a fixed simplicial complex $L$. As we will see, the Danzer and Davis complexes are closely related. This construction was first described explicitly by Danzer [10]. In Theorems 5.3.4 and 5.3.9, we prove a decomposition theorem for the homology of $\mathfrak{D}(L)$ and for pairs $(\mathfrak{D}(L), \mathfrak{D}(T))$ for an induced subcomplex $T \prec L$, and using this, we formulate the Charney-Davis conjecture as a statement about the minimal free resolution of the face ring of $L$ in Proposition 5.4.2.

**Definition 5.2.1.** Let $L$ be a simplicial complex on the vertex set $\{1, \ldots, n\}$, and $C_n$ the standard $n$-cube in $\mathbb{R}^n$. The *Danzer complex* $\mathfrak{D}(L)$ of $L$ is the subcomplex of $C_n$ which contains all faces of $C_n$ that are parallel to a affine subspace spanned by the coordinate axes corresponding to some $\sigma \in \mathcal{S}(L)$.

From now on, if a face $F$ of $\mathfrak{D}(L)$ is parallel to the hyperplane spanned by the coordinate axes corresponding to some $\sigma \in \mathcal{S}(L)$, we say that *$F$ is parallel to $\sigma$* or *$F$ is of type $\sigma$*. As the coordinates of $\mathbb{R}^n \supset C_n$ are determined by $V(L)$, we speak interchangeably of the coordinates of $\mathbb{R}^n \supset \mathfrak{D}(L)$ and the vertices of $V(L)$.

**Example 5.2.2.** Let $L$ consist of a vertex [3] and an edge [1 2]. Then $\mathfrak{D}(L)$ is the subcomplex of the cube in $\mathbb{R}^3$ pictured below:

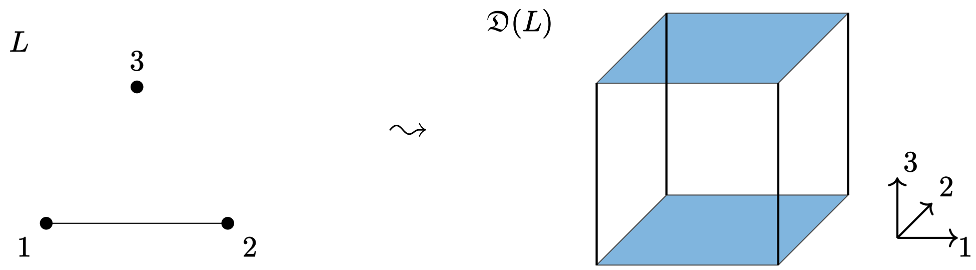


**Figure 2.3.** A simplicial complex $L$ on three vertices and the corresponding Danzer complex $\mathfrak{D}(L)$ as a subcomplex of the 3-cube.

The vertices of a link of a vertex $v \in \mathfrak{D}(L)$ are in bijection with $V(L)$; each vertex lies on an edge in the coordinate direction corresponding to some $v \in V(L)$. Then, a subset $\sigma$ of these vertices in the link span a simplex in the link if and only if the faces of type $\sigma$ are in $\mathfrak{D}(L)$. Hence, we see that each vertex link of $\mathfrak{D}(L)$ is isomorphic to $L$. The space $\mathfrak{D}(L)$ has a natural structure of an Euclidean cubical complex, and by Gromov's lemma 4.2.11 $\mathfrak{D}(L)$ is nonpositively curved (and consequentially aspherical by Theorem 4.2.7) if and only if $L$ is flag. As the vertex links are isomorphic to $L$, $\mathfrak{D}(L)$ is a closed $n$-manifold if and only if $L \cong \mathbb{S}^{n-1}$. Let us gather these main properties of $\mathfrak{D}(L)$ in the following proposition.

**Proposition 5.2.3.** *Let $L \subset [n]$ be a simplicial complex and $\mathfrak{D}(L)$ the corresponding Danzer complex. Then $\mathfrak{D}(L)$ is a finite Euclidean cube complex such that $lk_v(\mathfrak{D}(L)) \cong L$ for each vertex $v \in \mathfrak{D}(L)$. In particular, if $L$ is flag, then $\mathfrak{D}(L)$ is a nonpositively curved Euclidean cube complex, and if $L \cong \mathbb{S}^{n-1}$, then $\mathfrak{D}(L)$ is a closed $n$-manifold.*

The Danzer complex behaves similarly with respect to products as $\Sigma_L$, i.e., $\mathfrak{D}(L * K) \cong \mathfrak{D}(L) \times \mathfrak{D}(K)$.

To study $\mathfrak{D}(L)$ on the chain level, cell complexes are too coarse of a model for studying the homology of $\mathfrak{D}(L)$, and we prefer not to triangulate $\mathfrak{D}(L)$. A combinatorial model for cell complexes suitable for this situation are cell complexes formed by attaching polyhedra along their faces. A *convex $n$-polyhedron* is the convex hull of a finite set of points in $\mathbb{R}^n$. A *polyhedral complex* $\mathcal{K}$ is a collection of polyhedra such that each face of a polyhedron in $\mathcal{K}$ is also in $\mathcal{K}$, and the intersection of any two polyhedra $\sigma, \tau \in \mathcal{K}$ is a face of both $\sigma$ and $\tau$. For such a complex $\mathcal{K}$, we choose orientations for all individual polyhedra in $\mathcal{K}$. Then, we can form the *polyhedral chain complex* $(C_\bullet(\mathcal{K}), \partial)$ of $\mathcal{K}$, with $C_i(\mathcal{K})$ the $k$-vector space generated by the $i$-dimensional polyhedra of $\mathcal{K}$ and $\partial$ the polyhedral boundary operator, which is defined as follows; for an $i$-cell $\sigma \in \mathcal{K}$, we set $\partial\sigma = \sum_{\tau<\sigma} \operatorname{sign}(\tau,\sigma)\tau$, where the sum is over all the $(i-1)$-faces $\tau$ of $\sigma$, and $\operatorname{sign}(\tau,\sigma)$ equals $\pm 1$, depending on whether the orientation of $\tau$ agrees with the orientation inherited from $\sigma$. We extend this $k$-linearly to $C_i(\mathcal{K})$ to get the complex $(C_\bullet(\mathcal{K}), \partial)$. The homology of this chain complex of course agrees with the usual homology of the topological space $\mathcal{K}$.

Clearly, $\mathfrak{D}(L)$ is a polyhedral complex, where are cells are cubes of various dimension, and we use the polyhedral chain complex of $\mathfrak{D}(L)$ to compute its homology groups. For each vertex $v \in V(L)$ we have a well-defined action on $\mathfrak{D}(L)$ by reflecting along the $\{x_v = 0\}$ hyperplane, and hence we have an action of $W_L$ on $\mathfrak{D}(L)$, or more specifically of its abelianization. This action on $\mathfrak{D}(L)$ induces an action on the polyhedral chain complex $C_\bullet(\mathfrak{D}(L))$ as well as on the homology groups $H_*(\mathfrak{D}(L))$ of $\mathfrak{D}(L)$. For an element $w \in W_L$ and a chain $F = \sum_i \lambda_i F_i \in C_\bullet(\mathfrak{D}(L))$, we denote $w.F = \sum_i \lambda_i w.F_i$ for the reflected chain. For a subgroup $W_A \leq W_L$, we denote $W_A.F$ for the orbit of $F$ under the action of $W_A$.

If $L$ is a flag complex with an associated RACG $W_L$ and Davis complex $\Sigma_L$, then $\mathfrak{D}(L)$ has $\Sigma_L$ as its universal cover. In fact, $\mathfrak{D}(L)$ can be realized as the quotient of $\Sigma_L$ by the commutator subgroup $\Gamma_L$ of $W_L$, and $\Gamma_L$ is the fundamental group of $\mathfrak{D}(L)$ [11, p. 10-12]. Let $\widetilde{\mathfrak{D}}(\sigma)$ be the face of $\mathfrak{D}(L)$ parallel to $\sigma$ where all coordinates corresponding to $V(L \setminus \sigma)$ are equal to one. Altogether, we have the following proposition.

**Proposition 5.2.4.** *Let $L$ be a flag complex, $W_L$ the associated RACG, $\Gamma_L$ the commutator subgroup, and $\vartheta : W_L \to W_L^{ab}$ the natural epimorphism onto the abelianization. Then the map $\varphi : \Sigma_L \to \mathfrak{D}(L)$ sending the cube $w\Sigma_\sigma$ to $\vartheta(w).\widetilde{\mathfrak{D}}(\sigma)$ is a covering projection, and induces an $W_L$-equivariant homeomorphism $\overline{\varphi} : \Sigma_L/\Gamma_L \to \mathfrak{D}(L)$.*

In fact, any quotient of $\Sigma_L$ by a torsion-free subgroup of $W_L$ gives an aspherical space, and when $L$ is a flag sphere, these quotients are manifolds. As $\Sigma_L$ is contractible and the fundamental group of $\mathfrak{D}(L)$ is $\Gamma_L$, we see that $\mathfrak{D}(L)$ is a $K(\Gamma_L, 1)$ space, i.e., $\pi_1(\mathfrak{D}(L)) = \Gamma_L$ and $\pi_k(\mathfrak{D}(L)) = 0$ for all $k > 1$, as by Proposition 1.2.5 the higher homotopy groups of $\Sigma_L$

and $\mathfrak{D}(L)$ agree.

By Gromov's lemma 4.2.11, if $L$ is a flag complex, then $\mathfrak{D}(L)$ is nonpositively curved, which by the metric Cartan-Hadamard theorem guarantees that $\mathfrak{D}(L)$ is aspherical even without knowing the relation between $\mathfrak{D}(L)$ and $\Sigma_L$. As explained in Section 4.3, the Charney-Davis conjecture for a flag $(2k-1)$-sphere $L$ is equivalent to the Hopf conjecture for $\mathfrak{D}(L)$, i.e., that $(-1)^k\chi(\mathfrak{D}(L)) \geq 0$.

**Example 5.2.5.** Consider the flag triangulation $L$ of $\mathbb{S}^1$ as a 4-gon, which can be expressed as $L = \mathbb{S}^0 * \mathbb{S}^0$. Then $\mathfrak{D}(L) = \mathfrak{D}(\mathbb{S}^0) \times \mathfrak{D}(\mathbb{S}^0) \cong \mathbb{T}^2$, as can be seen from the cubical structure of $\mathfrak{D}(L)$, picture in Figure 2.4. The universal cover of the $n$-torus $\mathbb{T}^n$ is $\mathbb{R}^n$, so we see that $\mathfrak{D}(L)$ is aspherical. As expected, we have $(-1)^k\chi(\mathfrak{D}(L)) = 0 \geq 0$.

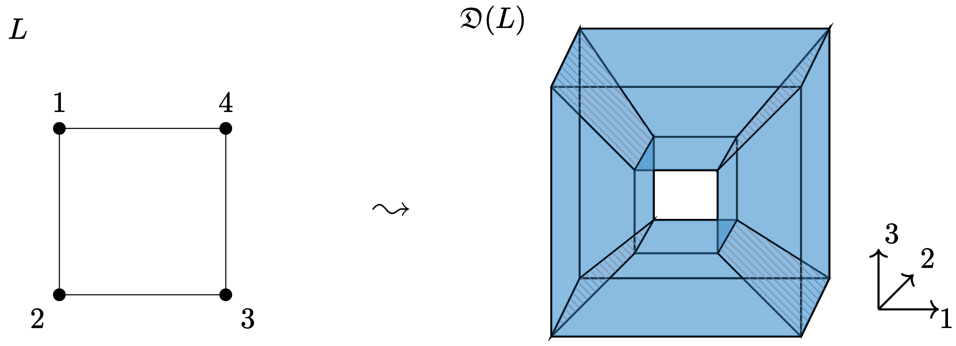


**Figure 2.4.** The Danzer complex of $L$ drawn in the Schlegel diagram of the 4-cube, projecting from the fourth coordinate. The patterned faces and those parallel to the plane $\{x_1 = 0, x_3 = 0\}$ are not in $\mathfrak{D}(L)$.

Similarly, we see that the octahedral flag $d$-sphere $L = \mathbb{S}^0 * \ldots * \mathbb{S}^0$, with $(d+1)$ copies of $\mathbb{S}^0$, has Danzer manifold $\mathfrak{D}(L) \cong \mathbb{S}^1 \times \ldots \times \mathbb{S}^1 = \mathbb{T}^d$, the $d$-torus.

For flag 1-spheres, the Charney-Davis conjecture can be verified by a simple combinatorial argument. Flag 1-spheres are $n$-gons with $n \geq 4$, with the $n = 4$ case resulting in $\mathfrak{D}(L) \cong \mathbb{T}^2$, so $(-1)^{\frac{n}{2}}\chi(D(L)) = -\chi(\mathbb{T}^2) = 0$. Any flag 1-sphere can be achieved by repeatedly subdividing edges of the 4-gon, and going from an $n$-gon $L_n$ to a $(n+1)$-gon $L_{n+1}$ by subdividing an edge increases the number of vertices of $\mathfrak{D}(L)$ by $2^n$, the number of edges by $2^{n-1}(n+1)$, and the number of faces by $2^{n-2}(n+2)$. Using the number of cells to compute the Euler characteristic of $L_{n+1}$, we have $\chi(L_{n+1}) = \chi(L_n)+2^n-2^{n-1}(n+1)+2^{n-2}(n+2) = \chi(L_n) + 2^{n-2}(4-n)$. As $n \geq 4$, this shows that the inequality $-\chi(\mathfrak{D}(L)) \geq 0$ is stable under the subdivision of edges, and hence the Charney-Davis conjecture holds for flag 1-spheres.

### 5.3 Homology of $\mathfrak{D}(L)$

The aim of this section is to prove Theorem 5.3.4, which allows us to compute the homology groups of $\mathfrak{D}(L)$ based on those of induced subcomplexes of $L$. In this section $L$ is an arbitrary simplicial complex on $n$ vertices.

For induced subcomplexes $\Delta \prec L$ of $L$, we have multiple ways to identify $\mathfrak{D}(\Delta)$ as a subcomplex of $\mathfrak{D}(L)$. For an induced subcomplex $\Delta \prec L$, we consider $\mathfrak{D}(\Delta)$ as a subcomplex of $\mathfrak{D}(L)$ by embedding $\mathfrak{D}(\Delta) \hookrightarrow \mathfrak{D}(L)$ to a homeomorphic copy of $\mathfrak{D}(\Delta)$ in $\mathfrak{D}(L)$. There are of course many different embeddings, corresponding to the reflections by the vertices of $L \setminus \Delta$, and for concreteness, we make the choice to identify $\mathfrak{D}(\Delta)$ with the copy where the coordinates corresponding to the vertices in $L \setminus \Delta$ are equal to 1. We denote $\widetilde{\mathfrak{D}}_L(\Delta)$ for the image of $\mathfrak{D}(\Delta)$ in $\mathfrak{D}(L)$ under this embedding, and leave the subscript out whenever the ambient complex is clear from context. We denote $W_L\widetilde{\mathfrak{D}}(\Delta)$ for the subcomplex of all faces parallel to the simplices of some $\Delta \prec L$, as this equals the $W_L$-orbit of $\widetilde{\mathfrak{D}}(\Delta)$. We speak interchangeably of the topological image of this embedding and the image on the chain level, i.e., of $\widetilde{\mathfrak{D}}(L)$ as a polyhedral chain in $C_\bullet(\mathfrak{D}(L))$. The chain $\widetilde{\mathfrak{D}}(\Delta)$ is the sum of all polyhedral faces parallel to some $\sigma \in \Delta$ whose $V(L \setminus \Delta)$-coordinates equal 1.

**Proposition 5.3.1.** *Let $\Delta, \Delta'$ be induced subcomplexes of $L$. Then*

$$\widetilde{\mathfrak{D}}(\Delta \cap \Delta') = \widetilde{\mathfrak{D}}(\Delta) \cap \widetilde{\mathfrak{D}}(\Delta')$$

*Proof.* If $F$ is a face of $\mathfrak{D}(\Delta \cap \Delta')$, then $F$ is of type $\sigma$ for some $\sigma \in \Delta \cap \Delta'$, and hence $F \in \mathfrak{D}(\Delta)$ and $F \in \mathfrak{D}(\Delta')$. Conversely, if $F$ is a face of $\mathfrak{D}(\Delta) \cap \mathfrak{D}(\Delta')$, then $F$ is of type $\sigma$ for some $\sigma$ with $\sigma \in \Delta$ and $\sigma \in \Delta'$, so $\sigma \in \Delta \cap \Delta'$ and hence $F \in \mathfrak{D}(\Delta \cap \Delta')$. □

We fix the orientations of the polyhedral faces of $\mathfrak{D}(L)$ to be such that the induced chain map $\widetilde{\mathfrak{D}}_\bullet : C_\bullet(L) \to C_\bullet(\mathfrak{D}(L))$ from the simplicial chain complex of $L$ to the polyhedral chain complex of $\mathfrak{D}(L)$ is a chain map in degrees $\geq 1$. Explicitly, if $\sigma = \{v_{k_1}, \ldots, v_{k_i}\} \subseteq L$ is a simplex in $L$ with $k_1 < \ldots < k_i$, then each boundary face $F$ of $\widetilde{\mathfrak{D}}(\sigma)$ is in the subcomplex $\widetilde{\mathfrak{D}}(\{v_{k_1}, \ldots, \hat{v_{k_j}}, \ldots, v_{k_i}\})$ for some $k_1 \leq j \leq k_i$. We choose the orientation of $F$ with respect to the $i$-face of $\widetilde{\mathfrak{D}}(\sigma)$ containing $F$ to be the sign of $\{v_{k_1}, \ldots, \hat{v_{k_j}}, \ldots, v_{k_i}\}$ appearing in the simplicial boundary formula of $\{v_{k_1}, \ldots, v_{k_i}\} = \sigma$. We choose the orientations of the edges of $\mathfrak{D}(L)$ to be such that $\partial(\widetilde{\mathfrak{D}}(\zeta)) = 0$ for any augmented 0-cycle $\zeta$ in $L$. Note that for any induced subcomplex $\Delta \prec L$ we get induced orientations from $L$, and we consider the chain complex of $\Delta$ as a subcomplex of $C_\bullet(L)$. With this choice of orientations we get the following.

**Proposition 5.3.2.** *The induced chain map* $\widetilde{\mathfrak{D}}_\bullet : (C_\bullet(L), \partial) \to (C_\bullet(\mathfrak{D}(L)), \partial)$*, which increases degree by one, is a chain map above degree 1, and maps (augmented) 0-cycles* $\zeta \in C_0(L)$ *to 1-cycles in* $C_1(\mathfrak{D}(L))$*.*

As usual, by $\widetilde{\mathfrak{D}}(\zeta) - \widetilde{\mathfrak{D}}(\zeta') = \widetilde{\mathfrak{D}}(\zeta - \zeta') = \widetilde{\mathfrak{D}}(\partial\tau) = \partial\widetilde{\mathfrak{D}}(\tau)$, this implies the following.

**Corollary 5.3.3.** *Homological equivalence in* $L$ *descends to homological equivalence in* $\mathfrak{D}(L)$*: if* $\zeta \sim \zeta' \in H_*(L)$ *with* $\zeta - \zeta' = \partial\tau$*, then* $\widetilde{\mathfrak{D}}(\zeta) \sim \widetilde{\mathfrak{D}}(\zeta')$ *with* $\widetilde{\mathfrak{D}}(\zeta) - \widetilde{\mathfrak{D}}(\zeta') = \widetilde{\mathfrak{D}}(\partial\tau) = \partial\widetilde{\mathfrak{D}}(\tau)$*.*

In particular, simplicial cycles in $L$ are mapped to polyhedral cycles in $\mathfrak{D}(L)$. Note that we may also work in characteristic 2 to circumvent any considerations on the orientations of faces in $\mathfrak{D}(L)$.

Furthermore, if $\zeta \subset L$ is an induced $(k-1)$-cycle, i.e., an induced subcomplex with nontrivial homology in degree $k-1$, then $\widetilde{\mathfrak{D}}(\zeta)$ is nontrivial in $H_k(\mathfrak{D}(L))$. We want to show that if we fix a basis $\{\zeta_j^\Delta\}$, $j = 1, \ldots, \dim \tilde{H}_{k-1}(\Delta)$, for the $(k-1)^{th}$ reduced homology of each induced subcomplex (or just induced cycles) $\Delta \prec L$, then certain reflections of the cycles $\widetilde{\mathfrak{D}}_\bullet(\zeta_j^\Delta)$ over all induced subcomplexes $\Delta$ and over all $j$ gives a basis for $H_k(\mathfrak{D}(L))$. In other words, the Danzer complexes of cycles of induced subcomplexes generate the homology of $\mathfrak{D}(L)$ to give the following decomposition.

**Theorem 5.3.4.** *Let* $L$ *be a* $(d-1)$*-dimensional simplicial complex and* $\mathfrak{D}(L)$ *the associated Danzer complex. Then*

$$H_k(\mathfrak{D}(L)) \cong \bigoplus_{\Delta \prec L} \tilde{H}_{k-1}(\Delta),$$

*where* $\Delta \prec L$ *denotes an induced subcomplex of* $L$*, including* $L$ *itself.*

Of course, the chain map $\mathfrak{D}_\bullet$ presented earlier does not give this induce this decomposition, as a generator $\zeta \in \tilde{H}_{k-1}(\Delta)$ for some $\Delta \prec L$ can also be a generator for some other induced subcomplex $\Delta'$. To get the correct embeddings $\tilde{H}_{k-1}(\Delta) \hookrightarrow H_k(\mathfrak{D}(L))$ for distinct induced subcomplexes, we require some dependence on the vertex set of $\Delta$.

To do so, we use the embedding $\widetilde{\mathfrak{D}}$ along with the subgroup $W_\Delta$ of $W_L$ generated by $V(\Delta)$. On the chain level this is given by the $k$-linear extension of the map $\widetilde{\mathfrak{D}}_\bullet^\Delta : (C_\bullet(L), \partial) \to (C_\bullet(\mathfrak{D}(L)), \partial)$ which sends a $k$-simplex $\sigma = [v_{i_1}, \ldots, v_{i_k}] \in \Delta$ to the sum of the $(k+1)$-faces

$F \in C_\bullet(\mathfrak{D}(L))$ contained in $\widetilde{\mathfrak{D}}(\Delta)$ parallel to $\sigma$; this equals the $W_\Delta$-orbit of $\widetilde{\mathfrak{D}}(\sigma)$ in $\mathfrak{D}(L)$. Denote this map by $\mathfrak{D}_\Delta$.

Now, we may use this map to embed the various $\tilde{H}_{k-1}(\Delta)$ into $H_k(\mathfrak{D}(L))$. First, fix a basis for $\tilde{H}_*(\Delta)$ for every induced subcomplex $\Delta \prec L$. For such an induced subcomplex $\Delta \prec L$ and a $k$-cycle $\mu \in \Delta$, define $\phi_\Delta(\mu)$ to be the $W_\Delta$-orbit of $\widetilde{\mathfrak{D}}(\mu)$, where by the $W_\Delta$-orbit we mean the orbit under the action of the subgroup $W_\Delta < W_L$ generated by $V(\Delta)$. Clearly we have $\partial\phi_\Delta(\sigma) = \phi_\Delta(\partial\sigma)$ for any $\sigma \in \mathcal{S}(\Delta)$, so $\phi_\Delta$ induces a well-defined map map $\phi_\Delta : \tilde{H}_{k-1}(\Delta) \to H_k(\mathfrak{D}(L))$.

**Lemma 5.3.5.** *The map $\phi_\Delta : \tilde{H}_{k-1}(\Delta) \to H_k(\mathfrak{D}(L))$ is injective for any induced subcomplex $\Delta \prec L$.*

*Proof.* We prove the claim by induction on $|V(\Delta \setminus \mu)|$ for any nontrivial $k-1$-cycle $\mu$ in $\Delta \prec L$. If $\Delta = \mu$, then $\phi_\Delta(\mu) = \widetilde{\mathfrak{D}}(\mu) \cong \mathfrak{D}(\mu)$, which is nonzero in $\tilde{H}_k(\mathfrak{D}(L))$. Proceeding inductively, suppose $\phi_\Delta(\mu) = 0$ in $H_k(\mathfrak{D}(L))$ for a $k-1$-cycle $\mu$ in $\Delta$. For any $v \in V(\Delta \setminus \mu)$, we have $\phi_\Delta(\mu) = W_\Delta.\widetilde{\mathfrak{D}}(\mu) = v.(W_{\Delta\setminus v}.\widetilde{\mathfrak{D}}(\mu)) + W_{\Delta\setminus v}.\widetilde{\mathfrak{D}}(\mu)$. By induction, $W_{\Delta\setminus v}.\widetilde{\mathfrak{D}}(\mu)$ is nontrivial in $H_k(\mathfrak{D}(L))$. Then, $v.(W_{\Delta\setminus v}.\widetilde{\mathfrak{D}}(\mu)) + W_{\Delta\setminus v}.\widetilde{\mathfrak{D}}(\mu)$ is a boundary in $\mathfrak{D}(L)$ if and only if $\mathfrak{D}(L)$ contains all $k+1$-faces of type $v \cup \sigma$, where $\sigma$ is a simplex of $L$ corresponding to the faces appearing in $W_{\Delta\setminus v}.\widetilde{\mathfrak{D}}(\mu)$. However, this implies that $v * \mu$ is in $\Delta$, and hence $\mu$ is trivial in $\tilde{H}_{k-1}(\Delta)$, as $\partial(v * \mu) = \mu$. □

First, to get an idea of why cycles in $\mathfrak{D}(L)$ are generated from induced cycles in $L$, let us first sketch how cycles in $\mathfrak{D}(L)$ can be decomposed into components for which the connection to induced cycles in $L$ is clearer. For a 1-cycle $\zeta$ in $\mathfrak{D}(L)$, fix some ordering of the vertices $\{v_i\}_{i=0}^m$ of the support $|\zeta|$ of $\zeta$. For $v_{i_0} \in |\zeta|$, consider the link of $\zeta$ in $\mathfrak{D}(L)$, i.e., in $K := \mathrm{lk}_{v_{i_0}}(\mathfrak{D}(L)) \cap |\zeta|$. For any two vertices $w, u$ in $K$, let $v_{i_1}, v_{i_2}$ be the vertices across the edges of $\mathfrak{D}(L)$ containing $w, u$ respectively, and $v_{i_3}$ the vertex across the $v_{i_1} - v_{i_2}$ diagonal to $v_{i_0}$. Then $[v_{i_0}v_{i_1}] + [v_{i_1}v_{i_3}] + [v_{i_3}v_{i_2}] + [v_{i_2}v_{i_0}]$ is, up to sign, a 1-cycle. Denote this cycle by $\eta$. Next, we define $\zeta'$ as the 1-cycle satisfying $\zeta = \zeta' + \eta$. Repeating this procedure for $\zeta'$, while still taking into account the vertices appearing in $|\eta|$, decomposes $\zeta$ into minimal components $\eta_\alpha$ which are all the polyhedral boundary of a square.

This process shows how to express $\zeta$ as sum of 1-cycles that are reflections of the standard embeddings $\widetilde{\mathfrak{D}}(\zeta) \subset \mathfrak{D}(L)$ for an induced augmented 0-cycle $\zeta$ of $L$. This idea generalizes naturally to the case of a $k$-cycles $\zeta$ in $\mathfrak{D}(L)$; again, look at the links of vertices of $\zeta$, and by the symmetry of $\mathfrak{D}(L)$ we can generate a cycle $\eta$ from the link, which we can use to decompose the cycle as $\zeta = \zeta' + \eta$. This procedure works analogously as in the 1-cycle case, giving a decomposition of the cycle into parts spanned by subcubes in $\mathfrak{D}(L)$. For such a decomposition of a cycle $\zeta \subset \mathfrak{D}(L)$, call the summands the *Danzer components* of $\zeta$. Note that this decomposition is not unique, and may depend on the choice of ordering of the vertices; for example, in Figure 2.5, the other three 1-cycles that are the faces of the 3-cube would also be valid (supports of) Danzer components of the cycle $\zeta$. However, we may fix an ordering of the vertices of $C_n \supset \mathfrak{D}(L)$, such as the lexicographic ordering with the ordering of the coordinates of $\mathbb{R}^n$ inherited from that of $V(L)$, to get a unique decomposition for a given polyhedral cycle $\zeta$ in $\mathfrak{D}(L)$. On the chain level, this representation is not independent of the choice of representative of the homology class $[\zeta] \in H_*(\mathfrak{D}(L))$, and some of the Danzer components of a cycle might be trivial in $H_*(\mathfrak{D}(L))$. For a polyhedral cycle $\zeta$ in $\mathfrak{D}(L))$, we choose some minimal representative of $[\zeta]$ in the sense of the support of the cycle before forming its Danzer decomposition; if $\zeta = \sum_i \lambda_i F_i$ with $\lambda_i = -\lambda_j$, $F_i = F_j$ for some $i \neq j$, remove these terms before decomposing $\zeta$.

With this choice of representation, if $\zeta = \partial F$ in $C_\bullet(\mathfrak{D}(L))$, then its Danzer components all have to be trivial in $H_*(\mathfrak{D}(L))$. Conversely, if $\zeta = \sum_\alpha \zeta_\alpha$ is a decomposition of a cycle into its Danzer components with $\zeta_\alpha = \partial F_\alpha$, then $\zeta = \sum_\alpha \partial F_\alpha = \partial(\sum_\alpha F_\alpha)$. Hence, we have the following lemma.

**Lemma 5.3.6.** *A cycle $\zeta$ is trivial in $H_*(\mathfrak{D}(L))$ if and only if its Danzer components are trivial in $H_*(\mathfrak{D}(L))$.*

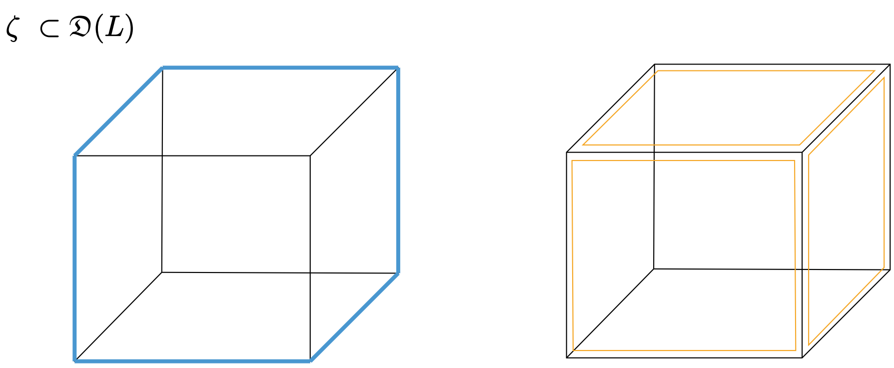


**Figure 2.5.** Decomposition of a cycle $|\zeta| \subset \mathfrak{D}(L)$ into (the supports of) its Danzer components.

Now, with a clearer picture of the structure of polyhedral cycles in $\mathfrak{D}(L)$, we may prove Theorem 5.3.4.

*Proof of Theorem 5.3.4.* First, let us show that the images of $\phi_\Delta$ for $\Delta \prec L$ generate the homology of $\mathfrak{D}(L)$. We again proceed inductively on $|V(L)| = n$; the reader may want to verify the cases where $n = 3$, where it is easy to see that the ranks of both sides of Theorem 5.3.4 coincide, and that the cycles $\phi_\Delta(\zeta)$ for $\Delta \prec L$ are linearly independent in $H_*(\mathfrak{D}(L))$. See Example 5.3.7 for how the decomposition works for the Danzer complex in Figure 2.3.

Now, let $\zeta$ be a nontrivial $k$-cycle in $\mathfrak{D}(L)$. Then, as in the case of 1-cycles, we inductively decompose $\zeta$ into its Danzer components, which are contained in the $(k+1)$-faces of $C_n$. If $k \leq n-2$, each component is contained in a face $C_{n-1} \subset C_n$ where some coordinate corresponding to a vertex $v_0 \in V(L)$ is equal to $\pm 1$. As $C_{n-1} \cap \mathfrak{D}(L) = \mathfrak{D}(L \setminus v)$, the result then follows by induction. As $|V(L)| = n$, the Danzer complex $\mathfrak{D}(L)$ is a subcomplex of $C_n$, and the only possible $(n-1)$-cycle $\zeta$ is $\partial C_n$. This is a cycle in $\mathfrak{D}(L)$ if and only if $L$ is either the $(n-1)$-simplex $\Delta^{n-1}$ or its boundary $\partial \Delta^{n-1}$. If $L = \Delta^{n-1}$, $\zeta = \partial C_n$ is of course trivial in $H_{n-1}(\mathfrak{D}(L))$, and if $L = \partial \Delta^{n-1}$, then $\zeta = \phi_L(L)$, where $L$ is considered as a simplicial cycle $H_{n-2}(L) \cong k$, and hence $H_{n-1}(\mathfrak{D}(L))$ is generated by $\phi_L(L)$.

Now, let us show that the images under $\phi_\Delta$ of the fixed basis elements of $\oplus_{\Delta \prec L} \tilde{H}_{k-1}(\Delta)$ are linearly independent in $H_k(\mathfrak{D}(L))$. For a collection of subcomplexes $\{\Delta_i\}_{i \in I}$ with $\tilde{H}_{k-1}(\Delta_i) \neq 0$ and basis elements $\{\zeta_{ij}\}_{j \in J} \in \tilde{H}_{k-1}(\Delta_i)$, denote $\phi_{\Delta_i}(\zeta_{ij}) := \xi_{ij}$, which equals the $W_{\Delta_i}$-orbit of $\widetilde{\mathfrak{D}}(\zeta_{ij})$. Note that we possibly have $\Delta_k = \Delta_l$ or $\zeta_{ki} = \zeta_{lj}$ for $k \neq l$.

Consider a linear combination $\sum_{i,j} \lambda_{ij} \xi_{ij}$ of such elements. If there exists some $v \in L \setminus \cup_i \Delta_i$, then the result follows by induction, as $\sum_{i,j} \lambda_{ij} \xi_{ij}$ is contained in $\mathfrak{D}(L \setminus v)$. Now, for the general case, as this linear combination is a cycle in $\mathfrak{D}(L)$, we may decompose it into its Danzer components $\eta_\alpha$. Again, if $k \leq n-2$, the components contained in each $C_{n-1}$-face of $C_n$ are not boundaries in $\mathfrak{D}(L)$ by induction. Furthermore, if the components on two opposing faces are homologous, i.e., $\eta_\alpha - \eta_\beta = \partial F$ in $C_\bullet(\mathfrak{D}(L))$, then all $(k+1)$-faces containing whose boundary on these faces is $\eta_\alpha$ and $\eta_\beta$ respectively are in $\mathfrak{D}(L)$, with $\eta_\alpha$ a reflection of $\eta_\beta$. As in the proof of Lemma 5.3.5, this would imply that one of the cycles in $\zeta_{ij}$ in $\Delta_i$ was trivial to begin with, or more precisely that $v * \zeta_{ij}$ is in $\Delta_i$, where $v$ is the vertex corresponding to the coordinate direction on which the cycles $\eta_\alpha, \eta_\beta$ lie. If $k = n-1$, the only possible case where $H_{n-1}(\mathfrak{D}(L))$ is nontrivial is when $L = \partial \Delta^{n-1}$ In this case $L$ itself is the only induced subcomplex with non-trivial $(n-2)$-homology, and $\phi_L(L)$ maps to the fundamental class in $H_{n-1}(\mathfrak{D}(L)) = H_{n-1}(\partial C_n)$.

□

**Example 5.3.7.**

(i) Consider the Danzer complex picture in Figure 2.3. The homologically nontrivial induced subcomplexes in $L$ are those on the vertices $\{1,3\}$, $\{2,3\}$ and $\{1,2,3\}$. Call these $K_1, K_2$ and $L$ respectively. Each of these induced subcomplexes has $\tilde{H}_0(\Delta) \cong k$. Fix the generators to be $\tilde{H}_0(K_1) = \langle [1]-[3] \rangle$, $\tilde{H}_0(K_2) = \langle [2]-[3] \rangle$ and $\tilde{H}_0(L) = \langle [1]-[3] \rangle$. The maps $\phi_{K_1}$ and $\phi_{K_2}$ map the generators of $\tilde{H}_0(K_1)$ and $\tilde{H}_0(K_2)$ to the square cycles on the $\{x_1 = 1\}$ and $\{x_2 = 1\}$ -faces of $\mathfrak{D}(L)$ respectively, and $\phi_L$ maps the generator $[1]-[3]$ of $\tilde{H}_0(L)$ to the orbit of $\widetilde{\mathfrak{D}}([1]-[3])$ under the action of $V(L)$, which is simply the sum of the square cycles on the $\{x_2 = 1\}$ and $\{x_2 = -1\}$ -faces of $\mathfrak{D}(L)$. These three 1-cycles are clearly linearly independent, and hence generate $H_1(\mathfrak{D}(L)) \cong k^3$.

(ii) When $L$ is the 4-gon, $\mathfrak{D}(L) \subset C_4$ is the 4-torus, as in Figure 5.2.5. This can also be seen from the fact that $L = \mathbb{S}^0 * \mathbb{S}^0$, so $\mathfrak{D}(L) = \mathfrak{D}(\mathbb{S}^0 * \mathbb{S}^0) = \mathfrak{D}(\mathbb{S}^0) \times \mathfrak{D}(\mathbb{S}^0) \cong \mathbb{S}^1 \times \mathbb{S}^1$. Then $H_1(\mathfrak{D}(L)) \cong k^2$ is generated by two 1-cycles along both copies $\mathbb{S}^1$, which correspond are the images of $\phi_{\Delta_0}$ and $\phi_{\Delta_1}$, where $\Delta_0$ is the induced subcomplex on $\{1,3\}$ and $\Delta_1$ that on $\{2,4\}$, for which we have $\tilde{H}_0(\Delta_0) = \langle [1]-[3] \rangle \cong k$ and $\tilde{H}_0(\Delta_0) = \langle [2]-[4] \rangle \cong k$.

**Corollary 5.3.8.** *If $L$ is a $(d-1)$-sphere, the sum of the Betti numbers of the induced cycles of $L$ satisfy*

$$\sum_{\Delta \prec L} \bar{b}_i(\Delta) = \sum_{\Delta \prec L} \bar{b}_{d-i-1}(\Delta),$$

*where $\bar{b}_i$ denote the reduced Betti numbers of $L$.*

*Proof.* If $L$ is a $(d-1)$-sphere, $\mathfrak{D}(L)$ is a compact $d$-manifold and hence satisfies Poincaré duality, with implies that $b_i(\mathfrak{D}(L)) = b_{d-i}(\mathfrak{D}(L))$. Then, the result follows immediately from Theorem 5.3.4. □

Let us briefly sketch another way to view this decomposition, which might be more natural given the main property of the Danzer complex $\mathfrak{D}(L)$, namely that all vertex links are isomorphic to $L$. Choose a vertex where we identify $L$ as the link, say $(1,\dots,1) \in C_n \supseteq \mathfrak{D}(L)$. Any vertex in the link lies on an edge in the coordinate direction corresponding to some $v \in V(L)$; we identify each vertex in the link with the corresponding vertex in $L$, and similarly for induced subcomplexes. Then, if $\Delta \prec L$ is an induced subcomplex and $\zeta$ is a $(k-1)$-cycle in $\tilde{H}_{k-1}(\Delta)$, we may generate a $k$-cycle $\phi_\Delta(\zeta)$ in $\mathfrak{D}(L)$ as the reflections under $V(\Delta)$ of the $k$-faces of $\mathfrak{D}(L)$ that correspond to the $(k-1)$-simplices in $\zeta$. See Figure 2.6 for an example.

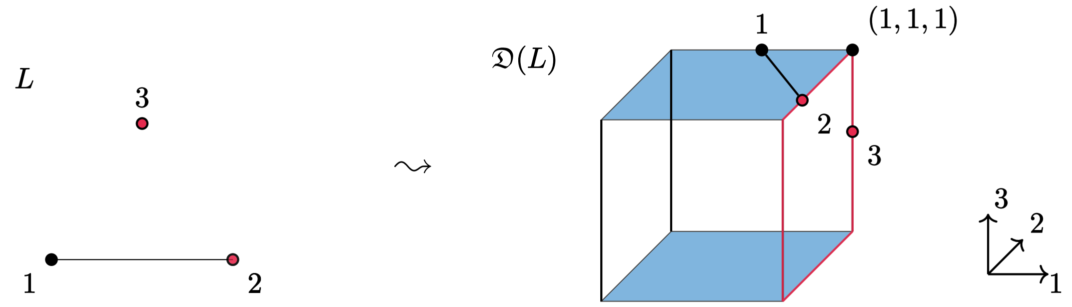


**Figure 2.6.** If $\Delta \prec L$ is the induced subcomplex on the vertices 2, 3, and $\zeta = [2] - [3]$ is a simplicial 0-cycle in $\Delta$, then the 1-cycle $\phi_\Delta(\zeta)$ generated by $\zeta$ is pictured in red.

Next, we prove a relative version of Theorem 5.3.4 for any induced subcomplex of $L$.

**Theorem 5.3.9.** *Let $T$ be an induced subcomplex of $L$ and $\{v_1, \dots, v_p\} = V(L) \setminus V(T)$ the complementary vertices in $L$. Denote $L_p = L \setminus \{v_1, \dots, v_p\}$. Then*

$$H_k(\mathfrak{D}(L), \mathfrak{D}(T)) = \Big(\bigoplus_{\substack{\Delta \prec L \\ v_1 \in \Delta}} \tilde{H}_{k-1}(\Delta)\Big) \oplus \Big(\bigoplus_{\substack{\Delta \prec L_1 \\ v_2 \in \Delta}} \tilde{H}_{k-1}(\Delta)\Big) \oplus \dots \oplus \Big(\bigoplus_{\substack{\Delta \prec L_{p-1} \\ v_p \in \Delta}} \tilde{H}_{k-1}(\Delta)\Big).$$

Here we view $\mathfrak{D}(T)$ as the image of any choice of embedding $\mathfrak{D}(T) \hookrightarrow \mathfrak{D}(L)$, for example as $\widetilde{\mathfrak{D}}(T)$. Before the proof of this result, let us recall some basic results on the homology of pairs and triples of spaces. Let $\iota : A \hookrightarrow X$ be a subspace of a topological space $X$. A *retraction* $r : X \to A$ is a continuous map such that $r(X) = A$ and $r\,|_A = \mathrm{id}_A$. Now, suppose $\iota : A \hookrightarrow X$ is a CW pair such that there exists a retraction $r : X \to A$. The induced map $\iota_* : H_i(A) \to H_i(X)$ is then injective, as $r_* i_* = \mathrm{id}_{H_i(A)}$. It follows from exactness that the boundary maps $\partial : H_{i+1}(X, A) \to H_i(A)$ in the long exact sequence of the pair $(X, A)$ are zero, and the long exact sequence breaks up in to short exact sequences

$$0 \longrightarrow H_i(A) \xrightarrow{\iota_*} H_i(X) \xrightarrow{j_*} H_i(X, A) \longrightarrow 0.$$

In fact, in this case even more is true, thanks to the following basic result in homological algebra.

**Lemma 5.3.10** (Splitting lemma [19, pp. 147])**.** *A short exact sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

*of abelian groups is equivalent to the sequence*

$$0 \longrightarrow A \xrightarrow{\ i\ } A \oplus C \xrightarrow{\ j\ } C \longrightarrow 0$$

*if and only if there exists a homomorphism $r : B \to A$ such that $r \circ i = id_A$.*

In our case, this tells us that if there exists a retract $r : X \to A$, the short exact sequences in homology split.

Now, consider the Danzer complex $\mathfrak{D}(L)$ of $L \subset [n]$ and a subcomplex $\widetilde{\mathfrak{D}}(L \setminus v)$ contained in a face $C_{n-1}$ of the $n$-cube $C_n$ of which $\mathfrak{D}(L)$ is a subcomplex. The orthogonal projection $C_n \to C_{n-1}$ onto this face containing $\widetilde{\mathfrak{D}}(L \setminus v)$ restricts to a retract $r : \mathfrak{D}(L) \to \mathfrak{D}(L \setminus v)$. We have an explicit description of the homology groups of $\mathfrak{D}(L)$ and $\mathfrak{D}(L \setminus v)$ using Theorem 5.3.4, and hence by the splitting of the short exact sequences of homology groups for the pair $(\mathfrak{D}(L), \mathfrak{D}(L \setminus v))$, we get

$$H_k(\mathfrak{D}(L), \mathfrak{D}(L \setminus v)) = \bigoplus_{\substack{\Delta \prec L \\ v \in \Delta}} \tilde{H}_{k-1}(\Delta)$$

For two vertices $v, w \in V(L)$, we get a triple of spaces $(\mathfrak{D}(L), \mathfrak{D}(L \setminus v), \mathfrak{D}(L \setminus \{v, w\}))$ and then by the retractions $r_v : \mathfrak{D}(L) \to \mathfrak{D}(L \setminus v)$ and $r_{vw} : \mathfrak{D}(L \setminus v) \to \mathfrak{D}(L \setminus \{v, w\})$, this sequence splits by the Splitting lemma 5.3.10. Using this, we may prove Theorem 5.3.9.

*Proof of Theorem 5.3.9.* For a vertex set $T$, fix the complementary vertices $\{v_1, \ldots, v_p\} \in V(L)$ with $p \leq n$. First, consider the triple $(\mathfrak{D}(L), \mathfrak{D}(L \setminus v_1), \mathfrak{D}(L \setminus \{v_1, v_2\})$. Identify $\mathfrak{D}(L \setminus \{v_1, v_2\})$ as usual as a the subcomplex of the $(n-2)$ face $C_{n-2}$ of $C_{n-1} \supset \mathfrak{D}(L \setminus \{v_1\})$ in the positive $v_2$-direction. We still have a retract $r : (\mathfrak{D}(L), \mathfrak{D}(L \setminus \{v_1, v_2\})) \to (\mathfrak{D}(L \setminus \{v_1\}), \mathfrak{D}(L \setminus \{v_1, v_2\}))$ given by the orthogonal projection, so by the injectivity of $i_*$ (which follows from $r_* i_* = \mathrm{id}$) and the Splitting lemma 5.3.10, the long exact sequence of this triple, i.e., the homology long exact sequence induced by

$$\begin{tikzcd} 0 \arrow[r] & C_\bullet(\mathfrak{D}(L \setminus \{v_1\}), \mathfrak{D}(L \setminus \{v_1, v_2\})) \arrow[r, "i"] & C_\bullet(\mathfrak{D}(L), \mathfrak{D}(L \setminus \{v_1, v_2\})) \arrow[dl, "\pi"] \\ & C_\bullet(\mathfrak{D}(L), \mathfrak{D}(L \setminus \{v_1\})) \arrow[r] & 0 \end{tikzcd}$$

breaks up into split short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & H_*(\mathfrak{D}(L\setminus\{v_1\}),\mathfrak{D}(L\setminus\{v_1,v_2\})) \arrow[r, "i_*"] & H_*(\mathfrak{D}(L),\mathfrak{D}(L\setminus\{v_1,v_2\})) \arrow[l, bend right, "r_*"'] \arrow[dl, "\pi_*"'] \\
& H_*(\mathfrak{D}(L),\mathfrak{D}(L\setminus\{v_1\})) \arrow[r] & 0.
\end{tikzcd}
$$

As

$$H_k(\mathfrak{D}(L\setminus\{v_1\}),\mathfrak{D}(L\setminus\{v_1,v_2\})) = \bigoplus_{\substack{\Delta\prec L\setminus\{v_1\}\\ v_2\in\Delta}} \tilde{H}_{k-1}(\Delta)$$

and

$$H_k(\mathfrak{D}(L),\mathfrak{D}(L\setminus\{v_1\})) = \bigoplus_{\substack{\Delta\prec L\\ v_1\in\Delta}} \tilde{H}_{k-1}(\Delta),$$

we get the required result by the splitting. Proceeding analogously with the triples $(\mathfrak{D}(L),\mathfrak{D}(L\setminus\{v_1,\ldots,v_{l-1}\}),\mathfrak{D}(L\setminus\{v_1,\ldots,v_l\}))$, we get the homology of the pair $P = (\mathfrak{D}(L\setminus\{v_1,\ldots,v_{l-1}\}),\mathfrak{D}(L\setminus\{v_1,\ldots,v_l\}))$, from the long exact sequence of $P$, and by induction we know the homology of $(\mathfrak{D}(L),\mathfrak{D}(L\setminus\{v_1,\ldots,v_{l-1}\}))$; then, the splitting of the homology short exact sequences of the triple $(\mathfrak{D}(L),\mathfrak{D}(L\setminus\{v_1,\ldots,v_{l-1}\}),\mathfrak{D}(L\setminus\{v_1,\ldots,v_l\}))$ gives the desired result. $\square$

**Remark 5.3.11.** For a CW pair $(X,A)$, if we define the relative Euler characteristic of $(X,A)$ as $\chi(X,A)=\sum_i(-1)^i\dim_k H_i(X,A)$, by the long exact sequence of the pair we have $\chi(X)=\chi(A)+\chi(X,A)$. Let $T\prec L$ be a codimension 2 induced flag sphere in a $(2k-1)$ flag sphere $L$, so that

$$(-1)^k\chi(\mathfrak{D}(L)) = (-1)^{k-2}\chi(\mathfrak{D}(T)) + (-1)^k\chi(\mathfrak{D}(L),\mathfrak{D}(T)).$$

By induction, we may assume $(-1)^{k-2}\chi(\mathfrak{D}(T))\geq 0$, which shows that the Charney-Davis conjecture for $L$ would follow from $(-1)^k\chi(\mathfrak{D}(L),\mathfrak{D}(T))\geq 0$ for any induced codimension 2 flag sphere $T$.

## 5.4 Applications

In this section we present some applications of the above results, especially to the Charney-Davis conjecture.

**Proposition 5.4.1.** *Let $L$ be a $(2k-1)$-dimensional flag simplicial sphere with $n$ vertices. Then the Charney-Davis conjecture for $\mathfrak{D}(L)$ is equivalent to*

$$(-1)^k\big(\sum_{\Delta\prec L}\tilde{\chi}(\Delta)\big)\geq 0,$$

*where $\tilde{\chi}$ is the reduced Euler characteristic, or in terms of the ordinary Euler characteristic*

$$(-1)^k\big(2^n-\sum_{\Delta\prec L}\chi(\Delta)\big)\geq 0.$$

*Proof.* The Euler characteristic of $\mathfrak{D}(L)$ is given by

$$\begin{aligned}\chi(\mathfrak{D}(L)) &= \sum_{i=0}^{2k}(-1)^i b_i(\mathfrak{D}(L)) = b_0-\sum_{i=1}^{2k}(-1)^{i+1}\big(\sum_{\Delta\prec L}\bar{b}_{i-1}(\Delta)\big)\\ &= 1-\sum_{\Delta\prec L}\tilde{\chi}(\Delta) = 1-\sum_{\Delta\prec L}(\chi(\Delta)-1)\\ &= 2^n-\sum_{\Delta\prec L}\chi(\Delta))\end{aligned}$$

□

One of the main methods in the modern study of the combinatorial properties of simplicial complexes is *Stanley-Reisner theory.* For a simplicial complex $\Delta$ on $[n]$, let $S=k[x_1,\dots,x_n]$ be a polynomial ring in $n$ variables over a field $k$. The *Stanley-Reisner* ideal of $\Delta$ is the ideal $I_\Delta\subset S$ generated by the monomials which correspond to non-faces of $\Delta$, i.e., $I_\Delta=\langle x_{i_1}\dots x_{i_k}\mid\{i_1,\dots,i_k\}\notin\Delta\}$. The *face ring* $k[\Delta]$ of $\Delta$ is then the quotient $S/I_\Delta$. This face ring has the usual structure of a graded ring, and the combinatorial properties of $\Delta$ are reflected in the algebraic structure of $k[\Delta]$. To study the properties of $k[\Delta]$, we may compute its minimal free resolution, which gives an exact sequence of free $S$-modules of finite length. From this resolution we can read of the *graded Betti numbers* $b_{i,j}(k[\Delta])$ of $k[\Delta]$, which are important algebraic invariants of graded modules. A result of Stanley [30] tells us that when $\Delta$ is a simplicial sphere, then $k[\Delta]$ is a *Gorenstein ring*, which implies certain symmetries in $k[\Delta]$ reminiscent of Poincaré duality for cohomology rings. For more details on face rings and Stanley-Reisner theory, we refer to [30, 27].

**Proposition 5.4.2.** *Let $\Delta$ be a $(2k-1)$-flag sphere on $n$ vertices, and let $S/I_\Delta=k[\Delta]$ be* *the face ring of $\Delta$. The Charney-Davis conjecture for $\mathfrak{D}(L)$ is equivalent to*

$$(-1)^k\left(\sum_{j=1}^{n}\sum_{i=0}^{j-1}(-1)^i b_{j-i-1,j}(k[\Delta])\right) \geq 0,$$

*where $b_{i,j}$ is the $(i,j)^{th}$ graded Betti number of $k[\Delta]$.*

*Proof.* We have

$$\begin{aligned}\sum_{\Delta\prec L}\tilde{\chi}(\Delta) &= \sum_{\Delta\prec L}\sum_{i=0}^{\dim\Delta}(-1)^i\tilde{H}_i(\Delta,k)\\ &= \sum_{j=1}^{n}\sum_{i=0}^{j}(-1)^i\sum_{|V(\Delta)|=j}\dim_k\tilde{H}_i(\Delta,k)\\ &= \sum_{j=1}^{n}\sum_{i=0}^{j}(-1)^i b_{j-i-1,j}(k[\Delta])\end{aligned}$$

Where the final equality follows from Hochster's formula ([27, Corollary 5.12]):

$$b_{i,j}(k[\Delta]) = \sum_{\substack{\Delta\prec L\\ |V(\Delta)|=j}}\dim_k\tilde{H}_{j-i-1}(\Delta,i).$$

The result then follows from Proposition 5.4.1. □

Hence, the Charney-Davis conjecture becomes a statement about the Betti table of the face ring of $\Delta$. As mentioned above, when $\Delta$ is a simplicial sphere, $k[\Delta]$ is a Gorenstein ring, and when $\Delta$ is flag, $I_\Delta$ is a quadratic monomial ideal, meaning that it is generated by monomials of the form $x_ix_j$, where $\{i,j\}$ is not an edge of $\Delta$.

# 6 $\ell^2$-invariants of the Davis complex

This section is dedicated to studying the $\ell^2$-homology of the Davis complex $\Sigma_L$. First, we cover the necessary background on cellular $\ell^2$-homology and -cohomology, and then prove the main results of Davis-Okun [12] on the Dodziuk-Singer conjecture for $\Sigma_L$. In particular, we prove that the Dodziuk-Singer conjecture for $\Sigma_L$ holds when $L$ is a flag $(n-1)$-dimensional generalized homology sphere for $n \leq 4$, which in turn yields a proof of the Charney-Davis conjecture for flag 3-spheres.

## 6.1 Cellular $\ell^2$-(co)homology

Next, we present the cellular approach to $\ell^2$-(co)homology. This material is from [25, 12] and [14], and proofs of all claims can be found in [25, Chapter 1].

Let $G$ be a discrete group. Throughout this section we will be working with real coefficients, but note that all could be done over $\mathbb{C}$ as well. Denote by $\ell^2(G)$ the Hilbert space of square-summable formal sums over $G$ with real coefficients, i.e., $\ell^2(G) = \{\sum_{g\in G} \lambda_g \cdot g \mid \sum |\lambda_g|^2 < \infty,\ \lambda_g \in \mathbb{R}\}$. We define a scalar product $\langle \cdot, \cdot \rangle$ on $\ell^2(G)$ by

$$\langle \sum_{g\in G} \lambda_g \cdot g, \sum_{g\in G} \mu \cdot g \rangle := \sum_{g\in G} \lambda_g \cdot \mu_g.$$

The group ring $\mathbb{R}G$ may be identified as the dense subspace of $\ell^2(G)$ consisting of finitely supported functions. The left action $l_h : \sum_{g\in G} \lambda_g \cdot g \mapsto \sum_{g\in G} \lambda_g \cdot hg$ is an orthogonal isometric $G$-action on $\ell^2(G)$. For a Hilbert space $H$, let $\mathcal{B}(H)$ be the algebra of bounded linear operators $H \to H$ with the operator norm $||A||_{op} := \{\sup \frac{||Av||}{||v||},\ v \neq 0,\ v \in H\}$.

**Definition 6.1.1** (Group von Neumann algebra)**.** The *group von Neumann algebra* $\mathcal{N}(G)$ of $G$ is the algebra of $G$-equivariant bounded operators on $\ell^2(G)$.

$$\mathcal{N}(G) := \mathcal{B}(\ell^2(G))^G.$$

Recall that for a subspace $W \subset \mathbb{R}^n$, the dimension of $W$ can be computed as the trace of the orthogonal projection $\mathrm{pr}_W : \mathbb{R}^n \to W$. In the case or group von Neumann algebras, we are dealing with operators on spaces which are possibly infinite-dimensional. However, we can still define a trace operator that allows us to extract some interesting information on the group von Neumann algebra.

**Definition 6.1.2** (Von Neumann trace)**.** The *von Neumann trace* on $\mathcal{N}(G)$ is defined by

$$\begin{aligned}\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) &\to \mathbb{R}\\ f &\mapsto \langle f(e), e\rangle_{\ell^2(G)},\end{aligned}$$

where $e \in G \subset \ell^2(G)$ is the unit element. Note that this is simply the coefficient of $e$ in $f(e)$.

The restriction of $\mathrm{tr}_{\mathcal{N}(G)}$ to the group algebra $\mathbb{R}G$ is the classical Kaplansky trace. The von Neumann trace has many of the properties of the usual trace from linear algebra, such as $\mathrm{tr}_{\mathcal{N}(G)}(f \circ g) = \mathrm{tr}_{\mathcal{N}(G)}(g \circ f)$ and $\mathrm{tr}_{\mathcal{N}(G)}(f) = \mathrm{tr}_{\mathcal{N}(G)}(f^*)$, where $f^*$ is the adjoint of $f$.

**Example 6.1.3.** If $G$ is finite, then $\ell^2(G) = \mathbb{R}G = \mathcal{N}(G)$, and we have

$$\begin{aligned}\mathrm{tr}_{\mathcal{N}(G)}(f) &= \langle f(e), e\rangle_{\mathbb{R}G}\\ &= \frac{1}{|G|}\sum_{g\in G}\langle f(g), g\rangle_{\mathbb{R}G}\\ &= \frac{1}{|G|}\mathrm{trace}(f),\end{aligned}$$

where $\mathrm{trace}(f)$ is the usual matrix trace of $f : \mathbb{R}G \to \mathbb{R}G$.

**Definition 6.1.4** (Hilbert $G$-modules)**.** A *Hilbert $G$-module* is a Hilbert space $V$ with a linear isometric $G$-action such that there exists a Hilbert space $H$ and an isometric linear $G$-embedding of $V$ into $H \otimes \ell^2(G)$, with $G$ acting on trivially on $H$. A *map of Hilbert $G$-modules* $f : V \to W$ is a bounded $G$-equivariant linear operator. $V$ is *finitely generated* if there exists a surjection $\oplus_{i=1}^n \ell^2(G) \to V$ of Hilbert $G$-modules for some $n \geq 0$.

A Hilbert $\mathcal{N}(G)$-module $V$ is finitely generated if and only if there is an isometric linear $G$-embedding $V \hookrightarrow \oplus_{i=1}^n \ell^2(G)$ for some $n \geq 0$. From now on, every Hilbert $G$-module is finitely generated.

If $H$ is a subgroup of $G$ and $V$ is a Hilbert $H$-module, the *induced representation* $\mathrm{Ind}_H^G(V)$ is the $\ell^2$-completion of $\mathbb{R}G \otimes_{\mathbb{R}H} W$, which is a Hilbert space with an orthogonal $G$-action. When $V$ is a closed subspace of $\ell^2(H)^n$, then $\mathrm{Ind}_H^G(V)$ is a closed subspace of $\ell^2(G)^n$, as $\mathrm{Ind}_H^G(\ell^2(H)) \cong \ell^2(G)$, so $\mathrm{Ind}_H^G(V)$ is a Hilbert $G$-module.

Recall that while the kernel of a continuous linear operator of Hilbert spaces is always closed, the image need not be. Hence, we have the following notion of exactness for sequences of maps of Hilbert $G$-modules.

**Definition 6.1.5.** A sequence of maps $V \xrightarrow{f} W \xrightarrow{g} U$ of Hilbert $G$-modules is *weakly exact* if the closure of the image of $f$ is the kernel of $g$. Similarly a map $f : V \to W$ is weakly surjective if the closure of the image equals $W$, and a weak isomorphism if it is furthermore injective.

**Definition 6.1.6.** Let $V$ be Hilbert $G$-module, and embed $V$ as a $G$-stable subspace of $\ell^2(G)^n$ for some $n \in \mathbb{N}$. Let $p_V : \ell^2(G)^n \to \ell^2(G)^n \supset V$ be the orthogonal projection onto $V$. Then the *von Neumann dimension* $\dim_G V$ of $V$ is

$$\dim_G V := \mathrm{tr}_{\mathcal{N}(G)}(p_V).$$

The von Neumann dimension is a nonnegative real number, and $\dim_G V = 0$ if and only if $V = 0$. For a finite group $G$, $\dim_G V = \frac{1}{|G|} \dim G$. We also have $\dim_G(\ell^2(G)^n) = n$.

Now, let us move on to the applications of these methods in topology. Let $G$ be a discrete group. A *$G$-complex* is a CW-complex $X$ together with a cellular action of $G$ on $X$. The complex $X$ is *geometric* if the $G$-action is proper and cocompact. Cocompactness is equivalent to $X$ having only finitely many $G$-orbits of cells in $X$, and properness means that the order of the stabilizer $G_\sigma$ of a cell $\sigma \in X$ is finite. From now on, we assume that every $G$-complex is geometric. A free $G$-complex is of course always proper. We are mainly interested in the case where $X$ is the universal cover of a compact CW-complex $Y$, where $\pi_1(Y)$ acts freely and properly on $X$ by deck transformations, and the quotient $X \to X/G$ is the covering projection.

If $Y$ is a subcomplex of $X$ stable under the action of $G$, then $(X, Y)$ is a *pair of $G$-complexes.* For a subgroup $H \leq G$ and a $H$-complex $X$, the *twisted product* $G \times_H X$ is the quotient space of $G \times X$ (where $G$ has the discrete topology) by the $H$-action $h.(g, x) = (gh^{-1}, hy)$. As $G/H$ is discrete, $G \times_H X$ is simply a disjoint union of copies of $X$ corresponding to the elements of $G/H$.

Given a $G$-complex $X$, we may regard the usual cellular chain complex $C_*(X) := C_*(X, \mathbb{R})$ as a complex of left $\mathbb{Z}G$-modules, with an element of $\phi \in C_i(X)$ being a finitely supported function on the $i$-cells of $X$. When $X$ is finite, $C_i(X)$ has a natural basis consisting of the $i$-cells $\{\sigma_j\}_{j \in I}$ of $X$, and we consider $C_i(X)$ as a real Hilbert space with the obvious Euclidean inner product $\langle \cdot, \cdot \rangle : C_i(X) \otimes C_i(X) \to \mathbb{R}$, making $\{\sigma_j\}_{j \in I}$ into an orthonormal basis. This gives a canonical isomorphism $C_i(X) \to C^i(X)$ by $\alpha \mapsto \langle \alpha, \cdot \rangle$. The usual boundary operator $d_i : C_i(X) \to C_{i-1}(X)$ has an adjoint $\delta_{i-1} : C_{i-1}(X) \to C_i(X)$ with respect to $\langle\cdot,\cdot\rangle$ such that $\langle d_i\alpha, \beta\rangle = \langle \alpha, \delta_{i-1}\beta\rangle$.

When $G$ acts cocompactly on $X$, each chain group $C_i(X)$ is a free $\mathbb{Z}G$-module whose rank equals the number of $i$-cells in $X/G$. Define the $\ell^2$ $i$-chains and cochains of $X$ as

$$C_i^{(2)}(X) = \ell^2(G) \bigotimes_{\mathbb{Z}G} C_i(X)$$
$$C^i_{(2)}(X) = \hom_{\mathbb{Z}G}(C_i(X), \ell^2(G)).$$

so that $C_i^{(2)}(X)$ is a left $\mathbb{R}G$-module. Hence, an element of $C_i^{(2)}$ can be regarded as an infinite chain with square-summable coefficients. As $\ell^2(G)$ can be identified with its dual, the modules $C_i^{(2)}(X)$ and $C^i_{(2)}$ are naturally isomorphic, and we will speak of these spaces interchangeably. For an $i$-cell $\sigma$ of $X$, the $\ell^2$-chains which are supported on the $G$-orbit of $\sigma$ can be identified with $\ell^2(G/G_\sigma)$, which is a Hilbert $G$-module as $G_\sigma$ is finite. As the action of $G$ is cocompact, there are finitely many orbits of cells in $X$, and hence $C_i^{(2)}(X)$ is a Hilbert $G$-module.

On the chain complex level the usual boundary $d_i : C_i(X) \to C_{i-1}(X)$ and coboundary $\delta^i : C^i \to C^{i+1}$ formulas induce $G$-equivariant boundary and coboundary maps $\ell^2(G) \otimes_G d_i : C^i_{(2)} \to C^{i-1}_{(2)}$, $\ell^2(G) \otimes_G \delta_i : C^i_{(2)} \to C^{i+1}_{(2)}$. We denote these induced maps also by $d$ and $\delta$. The fact that the boundary of an $\ell^2$-chain is square-summable follows from the following elementary result [14, Lemma 2.2.1]:

**Lemma 6.1.7.** *Let $f : (\mathbb{R}G)^n \to (\mathbb{R}G)^m$ be a map of $\mathbb{R}G$-modules. Then the induced map $\ell^2(G) \otimes_G f : \ell^2(G) \otimes_G (\mathbb{R}G)^n \cong (\ell^2(G))^n \to \ell^2(G) \otimes_G (\mathbb{R}G)^m \cong (\ell^2(G))^m$ is bounded.*

Hence we have $G$-equivariant, bounded linear maps, with the coboundary $\delta^i$ coinciding with the adjoint of $d_{i+1}$. Of course, these maps square to zero as usual. Let $Z_i^{(2)}$ and $Z^i_{(2)}$ be the kernel of $d_i$ and $\delta^i$ respectively, and similarly $B_i^{(2)}$ and $B^i_{(2)}$ the images of $d_{i+1}$ and $\delta^{i-1}$. The *reduced $\ell^2$-homology and -cohomology* are respectively defined as

$$H_i^{(2)}(X) = Z_i^{(2)}/\overline{B_i^{(2)}}$$
$$H^i_{(2)}(X) = Z^i_{(2)}/\overline{B^i_{(2)}}.$$

Quotienting by the closure has the effect that both of these spaces are Hilbert $G$-modules; they are Hilbert spaces (as a quotient by a closed subspace) and inherit the $G$-module structure from $C_i(X)$ and $C^i(X)$ respectively. As in the smooth case, one may of course also consider the quotients without taking closures, resulting in the *unreduced* $\ell^2$-(co)homology, but we are interested in the case where the quotients are Hilbert $G$-modules. An example of a square-summable 1-cycle is given in Figure 2.7.

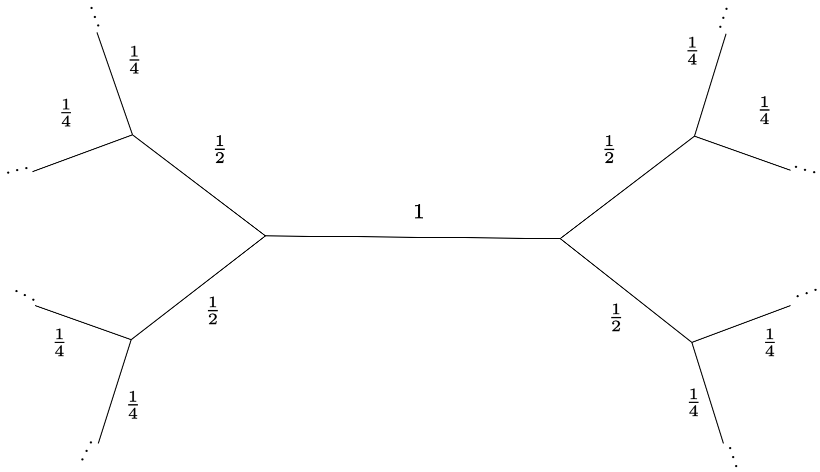


**Figure 2.7.** Example of an $\ell^2$ 1-cycle; the $\ell^2$-norm of this cycle is $1 + \sum_{k=0}^{\infty} 2^{k+1} \left(\frac{1}{2}\right)^{2n} = 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n-1} < \infty$.

As $\langle dx, y \rangle = \langle x, \delta y \rangle$, we have the orthogonal direct sum decompositions

$$C_i^{(2)}(X) = \overline{B_i^{(2)}} \bigoplus Z^i_{(2)}(X)$$
$$C^i_{(2)}(X) = \overline{B^i_{(2)}} \bigoplus Z_i^{(2)}(X).$$

Note that as $\langle dx, \delta y \rangle = \langle d^2 x, y \rangle = 0$, the subspaces $\overline{B_i^{(2)}}$ and $\overline{B^i_{(2)}}$ are orthogonal, which implies the cellular version of the $\ell^2$-Hodge decomposition

$$C_i^{(2)}(X) = \overline{B^i_{(2)}} \bigoplus \overline{B_i^{(2)}} \bigoplus (Z_i(X) \cap Z^i(X)),$$

from which it follows that both the $\ell^2$-homology and -cohomology groups can be identified with $Z_i(X) \cap Z^i(X)$. We denote this space by $\mathcal{H}_i(X)$, and refer to elements of $\mathcal{H}_i(X)$ as *harmonic i-cycles.* This comes from the fact that $Z_i(X) \cap Z^i(X)$ coincides with the kernel of the *combinatorial Laplacian* $\Delta = \delta^{i-1}\partial_i + \partial_{i+1}\delta^i : C_i^{(2)}(X) \to C_i^{(2)}(X)$.

The $\ell^2$-homology groups tend to be either zero or infinite-dimensional as Hilbert spaces. However, as in the case of analytic $\ell^2$-cohomology, more information can be extracted when taking into account the $G$-action.

**Definition 6.1.8** ($\ell^2$-Betti numbers). For a pair of geometric $G$-complexes $(X, Y)$, the $i^{th}$ $\ell^2$-Betti number $b_i^{(2)}(X, Y; G)$ is given by the von Neumann dimension

$$b_i^{(2)}(X, Y; G) := \dim_G(H_i^{(2)}(X, Y)).$$

The $\ell^2$-homology and -cohomology groups are $G$-equivariantly and isometrically isomorphic, so the $\ell^2$-Betti numbers can be computed as the von Neumann dimension of either module [25, pp. 34]. Of course, if $\dim X = n$, then $b_k^{(2)}(X) = 0$ for $k > n$.

As we are mostly concerned with the Dodziuk-Singer conjecture on the vanishing of certain $\ell^2$-Betti numbers, for us an important property of these invariants is that

$$b_i^{(2)}(X,Y;G) = 0 \quad \text{if and only if} \quad H_i^{(2)}(X,Y) = 0.$$

As in the case of the usual Euler characteristic, we define the *$\ell^2$-Euler characteristic* $\chi^{(2)}(X;G)$ of a $G$-complex $X$ as

$$\chi_{(2)}(X;G) = \sum_{i=0}^{\dim X} (-1)^i b_i^{(2)}(X;G)$$

We leave out $G$ from the notation for $b_i^{(2)}(X,Y;G)$ and $\chi^{(2)}(X;G)$ when it is clear from the context.

The orbihedral Euler characteristic $\chi_{\mathrm{Orb}}$ of $X/G$ is defined as

$$\chi_{\mathrm{Orb}}(X/G) := \sum_{\sigma \in G/X} \frac{(-1)^{\dim \sigma}}{|G_\sigma|}$$

where $\sigma \in G/X$ means we sum over representatives of the $G$-orbits of cells, and $|G_\sigma|$ is the order of the stabilizer of $\sigma$, which is of course independent of the choice of representative. If the $G$-action on $X$ is free, then $\chi_{\mathrm{Orb}}$ equals the ordinary Euler characteristic of $X/G$. If $H \leq G$ is a subgroup of finite index $[G : H] = k$, then $\chi_{\mathrm{Orb}}(X/H) = k\chi_{\mathrm{Orb}}(X/G)$.

Now, we list some basic properties of the $\ell^2$-(co)homology of pairs of $G$-complexes $(X,Y)$ [12, Section 2.4].

**Functoriality** Any $G$-equivariant map of pairs of $G$-complexes $f : (X_1, Y_1) \to (X_2, Y_2)$ induces a map $f_* : H_i^{(2)}(X_1, Y_1) \to H_i^{(2)}(X_2, Y_2)$, giving a functor from pairs of $G$-complexes to Hilbert $G$-modules. Moreover, if $f$ and $g$ are $G$-homotopic[1] maps, then $f_* = g_*$.

**Sequence of a pair** The long exact sequence of a pair of $G$-complexes $(X,Y)$

$$\dots \longrightarrow H_i^{(2)}(X) \longrightarrow H_i^{(2)}(Y) \longrightarrow H_i^{(2)}(X,Y) \longrightarrow \dots$$

[1]meaning that the homotopy between $f$ and $g$ is equivariant with respect to the $G$-action

is weakly exact.

**Excision** If $(X, Y)$ is a pair of $G$ complexes with $U \subset Y$ a $G$-stable subset such that $Y \setminus U$ is a subcomplex of $X$, then the inclusion $(X \setminus U, Y \setminus U) \to (X, Y)$ induces an isomorphism

$$H_i^{(2)}(X \setminus U, Y \setminus U) \xrightarrow{\sim} H_i^{(2)}(X, Y).$$

**Mayer-Vietoris** If $X = A \cup B$ for $G$-stable subcomplexes $A, B \subseteq X$, then $A \cap B$ is also $G$-stable and we have the Mayer-Vietoris sequence

$$\ldots \longrightarrow H_i^{(2)}(A \cap B) \longrightarrow H_i^{(2)}(A) \oplus H_i^{(2)}(B) \longrightarrow H_i^{(2)}(X) \longrightarrow \ldots$$

**Künneth Formula** Suppose $X$ is geometric $G$-complex and $Y$ is a geometric $H$-complex. Then $X \times Y$ is geometric $G \times H$-complex with

$$H_k^{(2)}(X \times Y) \cong \bigoplus_{i+j=k} H_i^{(2)}(X) \mathbin{\widehat{\bigotimes}} H_i^{(2)}(Y),$$

where $\widehat{\otimes}$ denotes the $\ell^2$-completion of the tensor product of Hilbert spaces.

**Poincaré duality** Let $X$ be an homology $n$-manifold with (possibly empty) boundary $\partial X$, and suppose that $(X, \partial X)$ is a pair of geometric $G$-complexes. Then

$$H_k^{(2)}(X, \partial X) \cong H_{(2)}^{n-k}(X) \quad \text{and}$$
$$H_k^{(2)}(X) \cong H_{(2)}^{n-k}(X, \partial X).$$

**Atiyah's formula**

$$\chi_{\mathrm{Orb}}(X/G) = \chi_{(2)}(X; G).$$

**Twisted products** If $H$ is a subgroup of $G$ and $X$ is a $H$-complex, then

$$H_k^{(2)}(G \times_H X) \cong \mathrm{Ind}_H^G(H_k^{(2)}(X)).$$

**Example 6.1.9.** An element of $C_{(2)}^i(X)$ is an $\ell^2$-function on $C_0(X) = V(X)$, and is a cocycle if and only if it has the same value on the endpoints of each edge. If $X$ is connected, any 0-cocycle must be constant, and hence $H_i^{(2)}(X) = 0$ if $X$ is a free $G$-complex for infinite $G$; no constant function on the vertices of an infinite complex $X$ is $\ell^2$. In fact, one can more generally show that for a connected free $G$-complex $X$, we have

$$b_0^{(2)}(X) = \frac{1}{|G|},$$

which we understand to be 0 when $G$ is infinite [25, Theorem 1.35].

**Example 6.1.10.** Let $S_g$ be the closed, orientable surface of genus $g \geq 1$. The fundamental group of $S_g$ is infinite, so by Example 6.1.9 we have $b_0^{(2)}(\widetilde{S_g}) = 0$, and by Poincaré duality $b_2^{(2)}(\widetilde{S_g}) = 0$. As $\chi(S_g) = 2 - 2g$, by Atiyah's formula 1.4.8 we get $b_1^{(2)}(\widetilde{S_g}) = 2g - 2$.

**Example 6.1.11.** Let $X$ be an $n$-dimensional $G$-pseudomanifold, i.e., a $G$-complex such that each $(n-1)$-cell is the face of exactly two $n$-cells. If a connected component of the complement of the $(n-2)$-skeleton of $X$ is not orientable, then $X$ has no nontrivial $n$-cycles, as in the case of non-orientable cellulated manifolds. If a component of the complement is orientable, then any $n$-cycle supported on this component is the sum of all chains in the component, with all coefficients equal, as every $(n-1)$-cell is contained in exactly two $n$-cells. Hence, if such a component has an infinite number of $n$-cells, it supports no $\ell^2$ $n$-cycle. From this we conclude that if the components of the complement of $X^{(n-2)}$ are either non-orientable or infinite, then $H_n^{(2)}(X) = 0$. This is in particular the case when the complement is connected and $G$ is infinite.

## 6.2 $\ell^2$-homology of $\Sigma_L$

As we have seen in Section 5, for any flag $(d-1)$-sphere $L$, we have an associated right-angled Coxeter group $W_L$ which acts properly and cocompactly on the Davis complex $\Sigma_L$. Any finite index torsion-free subgroup $\Gamma$ of $W_L$ acts properly and cocompactly on $\Sigma_L$, so that $\Sigma_L/\Gamma$ is a finite cubical complex. As $\Sigma_L$ is contractible $d$-manifold, $\Sigma_L/\Gamma$ is an aspherical $d$-manifold, with the Danzer complex $\mathfrak{D}(L)$ corresponding to the case where $\Gamma$ is the commutator subgroup. Hence, we get an ample supply of aspherical manifolds, and we may ask whether the Dodziuk-Singer conjecture holds in this case, i.e., if

$$b_i^{(2)}(\Sigma_L) = 0 \quad \text{for all} \quad i \neq \frac{d}{2}.$$

This conjecture was proved by Davis and Okun for $d \leq 4$ in [12] using Andreev's Theorem on three-dimensional hyperbolic polyhedra [2], or alternatively some powerful results from the theory of hyperbolic 3-manifolds explained at the end of this section, combined with inductive arguments on the dimension of $L$. In this section we will present the various versions of the Dodziuk-Singer conjecture for $\Sigma_L$ as well as some of the inductive arguments between these conjectures used in [12].

Let us fix some notation for convenience. For a flag complex $L$ and an induced subcomplex $A \prec L$, let

$$\begin{aligned} H_i^{(2)}(L) &= H_i^{(2)}(\Sigma_L) \\ H_i^{(2)}(A) &= H_i^{(2)}(W_L\Sigma_A) \\ H_i^{(2)}(L,A) &= H_i^{(2)}(\Sigma_L, W_L\Sigma_A), \end{aligned}$$

where $W_L\Sigma_A$ denotes the union of all translates of $\Sigma_A$ in $\Sigma_L$, i.e., the $W_L$-orbit of $\Sigma_A$ in $\Sigma_L$. We denote the corresponding $\ell^2$-Betti numbers and the $\ell^2$-Euler characteristic by

$$\begin{aligned} b_i^{(2)}(L) &= \dim_{W_L} H_i^{(2)}(L) \\ b_i^{(2)}(A) &= \dim_{W_L} H_i^{(2)}(A) \\ b_i^{(2)}(L,A) &= \dim_{W_L} H_i^{(2)}(L,A) \\ \chi^{(2)}(L) &= \sum_i (-1)^i b_i^{(2)}(L). \end{aligned}$$

Note that as $W_L\Sigma_A \cong W_L \times_{W_A} \Sigma_A$, we have $H_i^{(2)}(W_L\Sigma_A) = \mathrm{Ind}_{W_A}^{W_L} H_i^{(2)}(\Sigma_A)$.

**Example 6.2.1.**

**$k$-simplex** For a $k$-simplex $\Delta^k$, we have $W_{\Delta^k} \cong (\mathbb{Z})^{k+1}$ and $\Sigma_{\Delta^k} = [-1,1]^{k+1}$. As $\Sigma_{\Delta^k}$ is contractible and $W_{\Delta^k}$ is finite, we have $b_0^{(2)}(\Delta^k) = \frac{1}{|W_{\Delta^k}|} b_0(\Sigma_L) = \left(\frac{1}{2}\right)^{k+1}$ and $b_i^{(2)}(\Delta^k) = 0$ for $i \neq 0$.

**$k$ points** Let $P_k$ be the disjoint union of $k$ points. As $\Sigma_{P_k}$ is 1-dimensional, $b_i^{(2)}(P_k) = 0$ for $i \geq 2$. For $k \geq 0$, $b_0^{(2)}(P_k) = 0$, as $W_{P_k}$ is infinite. Then, by Atiyah's formula for the Euler characteristic 6.1, we have $b_1^{(2)}(P_k) = -\chi^{(2)}(P_k) = \frac{k}{2} - 1$. In particular, $b_i^{(2)}(\mathbb{S}^0) = 0$ for all $i$.

**Joins** As $\Sigma_{L_1 * L_2} = \Sigma_{L_1} \times \Sigma_{L_2}$, the Künneth formula gives $b_k^{(2)}(L_1 * L_2) = \sum_{i+j=k} b_i^{(2)}(L_1) b_j^{(2)}(L_2)$.

**Joins of $P_K$** If $L = P_{k_1} * \ldots * P_{k_m}$, with $k_j \geq 2$ for all $j$, then $b_m^{(2)}(L) = \prod_{j=1}^m (\frac{k_j}{2} - 1)$ and $b_i^{(2)}(L) = 0$ for all $i \neq m$.

**Suspension** As $b_i^{(2)}(\mathbb{S}^0) = 0$ for all $i$, by the formula for joins we have $b_i^{(2)}(\Sigma L) = b_i^{(2)}(\mathbb{S}^0 * L) = 0$ for all $i$.

**Disjoint union** If $L = L_1 \bigsqcup L_2$, then for $b_i^{(2)}(L) = b_i^{(2)}(L_1) + b_i^{(2)}(L_2)$ for $i \geq 2$, and if neither $L_1$ nor $L_2$ is a simplex, then $b_1^{(2)}(L) = b_1^{(2)}(L_1) + b_1^{(2)}(L_2) + 1$. This is a consequence of the $\ell^2$ Mayer-Vietoris sequence, noting that $b_0^{(2)}(L_1 \cap L_2) = b_0^{(2)}(\varnothing)$.

***k*-gon** Suppose $L$ is a $k$-gon with $k \geq 4$. Then $H_0^{(2)}(L) = 0$ as $W_L$ is infinite, and as $L \cong \mathbb{S}^1$, $\Sigma_L$ is a 2-manifold, so by Poincaré duality $H_2^{(2)}(L) = 0$. Using Theorem 5.3.4 we see that the complex $\mathfrak{D}(L)$ is a surface with Euler characteristic $2^{k-2}(4-k)$, and hence by Atiyah's formula for the Euler characteristic, $b_1^{(2)}(L) \neq 0$ if $k \geq 5$.

Now, let us connect some of the various invariants of $G$-complexes introduced earlier in the case of the Davis complex $\Sigma_L$, and see how they connect to the Charney-Davis conjecture.

**Lemma 6.2.2.** *The orbihedral Euler characteristic of $\Sigma_L/W_L$ is given by*

$$\chi^{orb}(\Sigma_L/W_L) = \sum_{k=-1}^{\dim L} \left(-\frac{1}{2}\right)^{k+1} f_k(L)$$

*Proof.* Each $(k+1)$-dimensional cube is in the $W_L$-orbit corresponding to some $\sigma \in \mathcal{S}(L)$, with the stabilizer of this cube being $W_\sigma \cong (\mathbb{Z}_2)^{k+1}$, which has order $2^{k+1}$. Hence we have $\chi^{\mathrm{orb}}(\Sigma_L/W_L) = \sum_{\sigma\in\mathcal{S}(L)} \left(-\frac{1}{2}\right)^{\dim\sigma+1}$, which equals the quantity in the Lemma. □

Note that this expression for the orbihedral Euler characteristic is exactly the Charney-Davis quantity $\kappa(L)$. Using Atiyah's formula 6.1 and Lemma 6.2.2, we get

$$\chi^{(2)}(L) = \sum_{i=-1}^{\dim L} \left(-\frac{1}{2}\right)^{i+1} f_i(L).$$

**Lemma 6.2.3.** *Let $CL$ denote the cone over $L$, and identify the Davis complex of $L$ as $\{\pm 1\} \times \Sigma_L$ in $\Sigma_{CL} = [-1,1] \times \Sigma_L$. Then we have the following.*

*(i) The sequence of the pair $(CL, L)$ breaks up into short exact sequences*

$$0 \longrightarrow H_{i+1}^{(2)}(CL, L) \longrightarrow H_i^{(2)}(L) \longrightarrow H_i^{(2)}(CL) \longrightarrow 0.$$

*(ii)* $b_{i+1}^{(2)}(CL, L) = b_i^{(2)}(CL) = \frac{1}{2} b_i^{(2)}(L)$.

*Proof.* As we are considering $\Sigma_L$ as $\{\pm 1\} \times \Sigma_L$ in the Davis complex of the pair $(CL, L)$, $H_i^{(2)}(L) = H_i^{(2)}(\Sigma_L) \oplus H_i^{(2)}(\Sigma_L)$ in the sequence of the pair $(CL, L)$. In the map induced by the inclusion $\{\pm 1\} \times \Sigma_L \hookrightarrow \Sigma_{CL}$, a class $(\zeta, -\zeta)$ maps to 0 in $H_i^{(2)}(CL)$ while the diagonal $(\zeta, \zeta)$ is mapped isomorphically to $H_i^{(2)}(CL)$, proving (i).

For (ii), we require two facts from the theory of Hilbert $G$-modules. Firstly, the von Neumann dimension of Hilbert $G$-modules behaves as expected for exact sequences, i.e., if $0 \to U \to V \to W \to 0$ is a short exact sequence of Hilbert $G$-modules, then $\dim_G V = \dim_G U + \dim_G W$ [25, Theorem 1.12]. Secondly, if $H$ is a subgroup of index $k$ in $G$, and $V$ is a Hilbert $G$-module, then $\dim_H V = k \dim_G V$ [25, Theorem 1.35].

As $CL = v * L$, we have $\Sigma_{CL} = [-1, 1] \times \Sigma_L$ and $W_{CL} = \mathbb{Z}_2 \times W_L$, with the $\mathbb{Z}_2$-factor acting by reflections on $[-1, 1]$. Hence, the homotopy contracting $[-1, 1] \times \Sigma_L$ to $\Sigma_L$ is $W_L$-equivariant, and as $W_L$ is an index 2 subgroup of $W_{CL} = \mathbb{Z}_2 \times W_L$, we have $b_i^{(2)}(CL) = \frac{1}{2} b_i^{(2)}(L)$. From $b_i^{(2)}(L) = b_{i+1}^{(2)}(CL, L) + b_i^{(2)}(CL)$ we get the desired result. $\square$

The inductive arguments of Davis and Okun are built around five following closely related conjectures $\mathsf{I} - \mathsf{V}(n)$ on the $\ell^2$-homology of $\Sigma_L$, where $L$ is a either a generalized homology sphere or disk and $n = \dim L + 1 = \dim \mathfrak{D}(L)$.

Instead of working with flag complexes which are topologically spheres, it in fact suffices to work with flag homology spheres, i.e., flag homology $n$-manifolds with the same same homology as $\mathbb{S}^n$. If $L$ is a flag homology $(n-1)$-sphere, then $\Sigma_L$ is a homology $n$-manifold, and in particular $\Sigma_L$ satisfies Poincaré duality 6.1, which states that $H_i^{(2)}(L) \cong H_{(2)}^{n-i}(L)$. Similarly, if $(D, \partial D)$ is a generalized homology $(n-1)$-disk, then $\Sigma_D$ is a homology manifold with boundary, and satisfies the relative version of Poincaré duality, $H_i^{(2)}(D) \cong H_{(2)}^{n-i}(D, \partial D)$.

From a generalized homology disk $(D, \partial D)$ we may form a homology sphere $S$ by coning over the boundary by a vertex $v$, such that the link of $v$ in $S$ is $\partial D$. Conversely, given a $\mathsf{GHS}$ $S$ and a vertex $v \in S$, we get a $\mathsf{GHD}$ $(D, \partial D)$ by deleting $v$ from $S$. We denote a $k$-dimensional generalized homology sphere by $\mathsf{GHS}^n$, and similarly for generalized homology disk.

### Davis-Okun conjectures

**$\mathsf{I}(n)$:** (Dodziuk-Singer conjecture): If $S$ is a $\mathsf{GHS}^{n-1}$, then $b_i^{(2)}(S) = 0$ for all $i \neq \frac{n}{2}$.

**$\mathsf{II}(n)$:** If $(D, \partial D)$ is a $\mathsf{GHD}^{n-1}$, then

- If $n = 2k$, then $b_i^{(2)}(D) = b_i^{(2)}(D, \partial D) = 0$ for all $i \neq k$.
- If $n = 2k+1$, then $b_i^{(2)}(D) = b_{i+1}^{(2)}(D, \partial D) = 0$ for all $i \neq k$, $b_k^{(2)}(D) = b_{k+1}^{(2)}(D, \partial D) = \frac{1}{2} b_k^{(2)}(\partial D)$ and the sequence

$$0 \longrightarrow H_{k+1}^{(2)}(D, \partial D) \longrightarrow H_k^{(2)}(\partial D) \longrightarrow H_k^{(2)}(D) \longrightarrow 0$$

is weakly exact.

$\mathsf{III}(2k+1)$**:** If $(D, \partial D)$ is a $\mathsf{GHD}^{2k}$ and $S = D \cup v * \partial D$ is the GHS formed by coning over the boundary, then in the Mayer-Vietoris sequence the map

$$H_k^{(2)}(D \cap CS_v) = h_k(S_v) \to H_k^{(2)}(D) \oplus H_k^{(2)}(CS_v)$$

is injective. This is equivalent to the map $H_k^{(2)}(S_v) \to H_k^{(2)}(S)$ induced by the inclusion being the zero map.

$\mathsf{IV}(2k+1)$**:** If $(D, \partial D)$ is a $\mathsf{GHD}^{2k}$, then $b_{k+1}^{(2)}(D) = 0$.

$\mathsf{V}(n)$**:** If $S$ is a $\mathsf{GHS}^{n-1}$ and $A$ is a full subcomplex, then

- $b_i^{(2)}(S, A) = 0$ for all $i > k$ if $n = 2k$ is even, and
- $b_i^{(2)}(A) = 0$ for all $i > k$ if $n = 2k+1$ is odd.

**Lemma 6.2.4.** *Let $J$ stand for any of the above statements. If $\mathsf{J}(n)$ holds for some $n \geq 2$, then so does $\mathsf{J}(n-2)$.*

*Proof.* Suppose for a contradiction that $\mathsf{J}(n)$ holds and $\mathsf{J}(n-2)$ fails for some $K$, a GHS or GHD of dimension $n-3$. If $L_5$ denotes the 5-gon, then as $H_1^{(2)}(L_5) \neq 0$, the Künneth formula for the $\ell^2$-cohomology 6.1 of $\Sigma_{L_5 * K} = \Sigma_{L_5} \times \Sigma_K$ shows that $\mathsf{J}(n)$ fails for $L_5 * K$. □

Let us list some easy implications between these conjectures.

**Lemma 6.2.5.**

*(i)* $\mathsf{II}(n) \Rightarrow \mathsf{I}(n-1)$

*(ii)* $\mathsf{II}(2k) \Rightarrow \mathsf{I}(2k)$

*(iii)* $\mathsf{I}(2k+1) \Rightarrow \mathsf{III}(2k+1)$

*(iv)* $\mathsf{II}(2k+1) \Rightarrow \mathsf{IV}(2k+1)$

(v) $\mathbf{V}(n) \Rightarrow \mathbf{I}(n)$.

*Proof.* For (i), if $\mathbf{II}(n)$ holds for the $\mathsf{GHD}^{n-1}$ $(CS, S)$ for a $\mathsf{GHS}^{n-2}$ $S$, $\mathbf{I}(n-1)$ for $S$ follows from Lemma 6.2.3.

For (ii), let $S = D \cup CS_v$ be a $\mathsf{GHS}^{2k-1}$ for some vertex $v \in S$. If $\mathbf{II}(2k)$ holds, by (i) $\mathbf{II}(2k-1)$ holds for the link $S_v$, i.e., $H_*^{(2)}(S_v) = 0$. Using the Mayer-Vietoris sequence for $S = D \cup CS_v$, we get $H_i^{(2)}(S) \cong H_i^{(2)}(D, S_v) \oplus H_i^{(2)}(CS_v, S_v)$, which both vanish by $\mathbf{II}(2k)$ for all $i \neq k$, and hence $\mathbf{I}(2k)$ holds for $S$.

Statement (iii) follows from the Mayer-Vietoris sequence in $\mathbf{III}(2k+1)$, and (iv) is immediate. For part (v), if $n = 2k$, taking $A = \varnothing$ and applying Poincaré duality shows that $H_i^{(2)}(S) = 0$ for all $i \neq k$, and if $n = 2k+1$ taking $A = S$ proves $\mathbf{I}(2k+1)$, again by Poincaré duality. $\square$

Next, we want to study some further connections between the above statements. To do so, let us fix some notation. Throughout the rest of this section, $S$ denotes a generalized homology sphere, $A \prec S$ is an induced subcomplex, and $B = A - v$ for some vertex $v \in A$. Then $A = B \cup C\mathrm{lk}_v A$ by joining $v$ to the vertices in $B$ adjacent to $v$, so $B \cap C\mathrm{lk}_v A = \mathrm{lk}_v A$.

**Lemma 6.2.6.** *The statement $\mathbf{V}(2k)$ implies that for any induced subcomplex $A \prec S$, where $S$ is a $\mathsf{GHS}^{2k-1}$, $H_i^{(2)}(A) = 0$ for all $i > k$.*

*Proof.* If $\mathbf{V}(2k)$ holds, then $H_i^{(2)}(S) = 0$ for all $i \neq k$ as explained above, and hence in the sequence $H_{i+1}^{(2)}(S, A) \to H_i^{(2)}(A) \to H^{(2)}(S)$ induced by the pair $(S, A)$, the first and third term vanish. $\square$

**Lemma 6.2.7.** *$\mathbf{V}(2k-1)$ implies $\mathbf{V}(2k)$.*

*Proof.* Suppose $\mathbf{V}(2k-1)$ holds, and let $S$ be a $\mathsf{GHS}^{2k}$ and $A \prec S$ an induced subcomplex. Let $B = A - v$ for some vertex $v \in B$. By induction on $|\mathbf{V}(S \setminus A)|$, assume that $\mathbf{V}(2k)$ holds for $(S, A)$, with the base case $A = S$ being trivial. The triple $(S, A, B)$ gives rise to the long exact sequence

$$\ldots \longrightarrow H_i^{(2)}(A, B) \longrightarrow H_i^{(2)}(S, B) \longrightarrow H_i^{(2)}(S, A) \longrightarrow \ldots,$$

and by induction $H_i^{(2)}(S, A) = 0$ for $i > k$. Next, consider the pair $(C\mathrm{lk}_v A, \mathrm{lk}_v A)$, formed by coning the link of $v \in A$. By excision, we have $H_i^{(2)}(A, B) = H_i^{(2)}(C\mathrm{lk}_v A, \mathrm{lk}_v A)$, which by Lemma 6.2.3 vanishes if and only if $H_{i-1}^{(2)}(\mathrm{lk}_v A) = 0$, which follows from the assumption that $\mathbf{V}(2k-1)$ holds, as $i > k$. Hence also $H_i^{(2)}(S, B) = 0$. $\square$

**Lemma 6.2.8.** *If* $\mathbf{V}(2k)$ *and* $\mathbf{III}(2k+1)$ *hold, then so does* $V(2k+1)$.

*Proof.* Let $S$ be a $(2k+1)$ GHS and $A, B$ as before. We have the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_i^{(2)}(B) \oplus H_i^{(2)}(C\mathrm{lk}_v A) \longrightarrow H_i^{(2)}(A) \longrightarrow H_{i-1}^{(2)}(\mathrm{lk}_v A) \longrightarrow \cdots$$

By induction on $|V(B)|$, we may assume $\mathbf{V}(2k+1)$ for $B \prec S$, and we want to show that $H_i^{(2)}(A) = 0$ for $i > k$. If $i \geq k+2$, then by Lemma 6.2.6, $H_{i-1}^{(2)}(\mathrm{lk}_v A) = 0$ and hence $H_i^{(2)}(C\mathrm{lk}_v A) = 0$ by Lemma 6.2.3. By induction $H_i^{(2)}(B) = 0$, and hence all terms in the Mayer-Vietoris sequence vanish, showing that $H_i^{(2)}(A) = 0$.

For the $i = k+1$ case, consider the Mayer-Vietoris sequence for the GHD $D = S - v$ with $S = D \cup C\mathrm{lk}_v S$ and the exact sequence of the pair $(\mathrm{lk}_v S, \mathrm{lk}_v A)$, compared with the Mayer-Vietoris sequence for $A = B \cup C\mathrm{lk}_v A$. Consider the diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & \\
 & & H_{k+1}^{(2)}(\mathrm{lk}_v S, \mathrm{lk}_v A) \arrow[d] & & \\
\cdots \arrow[r] & H_{k+1}^{(2)}(A) \arrow[r, "\partial_*"] & H_k^{(2)}(\mathrm{lk}_v A) \arrow[r, "j_* \oplus h_*"] \arrow[d, "\iota_*"] & H_k^{(2)}(B) \oplus H_k^{(2)}(C\mathrm{lk}_v A) \arrow[r] \arrow[d, "f_* \oplus g_*"] & \cdots \\
\cdots \arrow[r] & H_{k+1}^{(2)}(S) \arrow[r] & H_k^{(2)}(\mathrm{lk}_v S) \arrow[r, "k_* \oplus l_*"] \arrow[d] & H_k^{(2)}(D) \oplus H_k^{(2)}(C\mathrm{lk}_v S) \arrow[r] & \cdots \\
 & & \vdots & &
\end{tikzcd}$$

The space mapping into $H_{k+1}^{(2)}(A)$ is $H_{k+1}^{(2)}(B) \oplus H_{k+1}^{(2)}(C\mathrm{lk}_v A)$, which is trivial as in the $i \geq k+2$ case, so the map $\partial_*$ is injective. The maps $s_* \oplus t_*$ are induced by the inclusions $s : B \hookrightarrow D$ and $t : C\mathrm{lk}_v A \hookrightarrow C\mathrm{lk}_v S$. By assumption $\mathbf{V}(2k)$ holds, so $H_{k+1}^{(2)}(\mathrm{lk}_v S, \mathrm{lk}_v A) = 0$, so $\iota_*$ is injective. By $\mathbf{III}(2k+1)$, the map $k_* \oplus l_*$ is injective, which by commutativity of the square on the right implies that $j_* \oplus h_*$ is injective. This implies that the map $\partial_*$ is trivial, and hence $H_{k+1}^{(2)}(A) = 0$ by exactness. □

Of the statements $\mathbf{I} - \mathbf{V}$, $\mathbf{I}(n)$ is the usual Dodziuk-Singer conjecture, and $\mathbf{V}(n)$ is the strongest statement. The main result of this section is is the following:

**Theorem 6.2.9.** *Conjecture* $\mathbf{III}(2k-1)$ *implies* $\mathbf{V}(n)$ *for all* $n \leq 2k$.

*Proof.* Suppose inductively that $\mathbf{V}(n-1)$ holds for $n \leq 2k$. By Lemma 6.2.4, $\mathbf{III}(2k-1)$

implies $\mathbf{III}(2l-1)$ for all $l \leq k$. If $n-1$ is odd, then by Lemma 6.2.7 $\mathbf{V}(n)$ holds, and if $n-1$ is even, then by Lemma 6.2.8, $\mathbf{III}(n)$ and $\mathbf{V}(n-1)$ imply that $\mathbf{V}(n)$ holds. □

In [26], Lott and Lück prove the Dodziuk-Singer conjecture for closed, irreducible 3-manifolds with infinite fundamental group which satisfy Thurston's Geometrization conjecture. The conjecture for so-called Haken manifolds is implied by Thurston's hyperbolization theorem, and as is explained in [12, Sections 10.1.5, 14.1.6], if $S$ is a flag 2-sphere, then $\mathfrak{D}(S)$ is Haken, and hence the $\ell^2$-homology of $\Sigma_S = \widetilde{\mathfrak{D}(S)}$ vanishes. Davis and Okun also prove this fact directly using Andreev's theorem [2] on convex polyhedra in hyperbolic 3-space. As these results on 3-manifolds are outside the scope of this thesis, we refer to [12, Section 10] and the references therein for fruther details.

Combining this result with the inductive arguments presented in this section yields the following.

**Theorem 6.2.10.** *The statement $\mathbf{V}(n)$, and hence the Dodziuk-Singer conjecture for $\Sigma_L$, holds for $n \leq 4$.*

*Proof.* As explained above, the statement $\mathbf{I}(3)$ holds by the Lott-Lück theorem. As $\mathbf{I}(3)$ implies $\mathbf{III}(3)$, by Theorem 6.2.9 we get that $\mathbf{V}(4)$ holds. □

Hence, we get a proof of the Charney-Davis conjecture for flag 3-spheres, and as a consequence a proof of the Hopf conjecture for nonpositively curved manifolds cellulated by Euclidean cubes.

**Theorem 6.2.11.** *Let $L$ be a flag triangulation of a homology 3-sphere. Then*

$$\kappa(L) = 1 + \sum_{i=0}^{3} \left(-\frac{1}{2}\right)^{i+1} f_i(L) \geq 0.$$

*In particular, if $M$ is a nonpositively curved 4-manifold cellulated by Euclidean cubes, then*

$$\chi(M) \geq 0.$$

# Conclusions and further work

In this thesis we studied the Hopf and Dodziuk-Singer conjectures for spaces of nonpositive curvature. In Chapter 1, we presented proofs of the Dodziuk-Singer conjectures for Kähler hyperbolic manifolds and rotationally symmetric manifolds, including hyperbolic space. Chapter 2 was dedicated to studying the conjectures in a combinatorial context, especially focusing on the Charney-Davis conjecture on the face enumeration of flag spheres. Using the Danzer complex $\mathfrak{D}(L)$ construction for a simplicial complex $L$, we proved a decomposition theorem relating the homology of $\mathfrak{D}(L)$ to the reduced homology of induced subcomplexes of $L$. Using this and a relative version of this decomposition, we presented equivalent versions of the Charney-Davis conjecture in both a topological and purely algebraic context. Finally, we studied the Dodziuk-Singer conjecture for the universal cover of $\mathfrak{D}(L)$, concluding in a proof of the Charney-Davis conjecture for flag 3-spheres.

As we have seen during this thesis, there are many interesting questions yet to be answered in this topic. In the case of smooth manifolds, the classes of spaces for which the Hopf and Dodziuk-Singer conjectures are known are still relatively few, and all the known cases depend on strong geometric assumptions on the spaces involved. Especially the Hopf-Thurston conjecture, which extends the Hopf conjecture beyond the realm of geometry, seems very far out of reach with current methods. In the combinatorial case, the Charney-Davis conjecture remains open, as well as its generalization by Gal, which we presented in Section 4. Similarly, the vanishing of $\ell^2$-homology of $\Sigma_L$ outside the middle dimension remains unknown in dimension greater than 4. As the result of Davis and Okun relied heavily on results on 3-manifolds, new techniques are likely required to approach the generel case.

An interesting question on the Danzer complex construction is to study the structure of the cohomology ring of $\mathfrak{D}(L)$ for flag-triangulated pseudomanifolds $L$. Using intersection products in cohomology, combined with the decomposition results of Section 5, one can study the structure of cup products in $H^*(\mathfrak{D}(L))$ as intersections of induced cycles in $L$.

Here, some results are already known; a result of Athanasiadis [3] tells us that if $S$ is a flag-triangulated surface, then any induced 0-cycle in $S$ is the transverse intersection of two induced 1-cycles in $S$. Using a duality between intersections and cup products, this can be used to prove that the the cohomology ring of $\mathfrak{D}(S)$ is generated in degree one, i.e., any 2-cocycle is the cup product of 1-cocycles. Extending the result of Athanasiadis to higher dimensions, i.e., showing that induced 0-cycles in a flag-triangulated manifold $L$ are transverse intersections of codimension 1 cycles, would imply that $H^*(\mathfrak{D}(L))$ is generated in degree one.

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